MAA Metro NY Problem of the Month.

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Problem 1. *Compute*

$$\lim_{n\to+\infty} n\log\left(\sum_{k=1}^{n}\frac{1}{\binom{n}{k}}\right).$$

Proof. For $n \geq 6$, we have

$$\log\left(\sum_{k=1}^{n}\frac{1}{\binom{n}{k}}\right) = \log\left(1+\frac{2}{n}+\frac{4}{n(n-1)}+\sum_{k=3}^{n-3}\frac{1}{\binom{n}{k}}\right).$$

For all $x > 0$, we have

$$x-\frac{x^2}{2} \leq \log(1+x) \leq x.$$

This is easily seen by letting $f(x) = x - \frac{x^2}{2} - \log(1+x)$, then $f^{'}(x) = -x/(1+x) < 0$ for $x > 0$ and since $f(0) = 0$, it readily follows that $x - \frac{x^2}{2} \leq \log(1+x)$. The other direction, $\log(1+x) \leq x$ is trivial. As a consequence, we get

$$\frac{2}{n}-\frac{2}{n^2} \leq \log\left(1+\frac{2}{n}\right) \leq \log\left(1+\frac{2}{n}+\frac{4}{n(n-1)}+\sum_{k=3}^{n-3}\frac{1}{\binom{n}{k}}\right) \leq \frac{2}{n}+\frac{4}{n(n-1)}+\sum_{k=3}^{n-3}\frac{1}{\binom{n}{k}}.$$

On the other hand, for all $k \in [3, n-3]$, we have

$$\binom{n}{k} \geq \binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

Thus,

$$\sum_{k=3}^{n-3}\frac{1}{\binom{n}{k}} \leq \frac{6}{(n-1)(n-2)}.$$

Therefore,

$$2-\frac{2}{n} \leq n\log\left(\sum_{k=1}^{n}\frac{1}{\binom{n}{k}}\right) \leq 2+\frac{4}{n-1}+\frac{6n}{(n-1)(n-2)}.$$

Taking n to infinity, we get

$$\lim_{n\to+\infty} n\log\left(\sum_{k=1}^{n}\frac{1}{\binom{n}{k}}\right) = 2.$$

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