

Solution by Dr. Anna Pun

(a) Consider the generating function for the total number of points:

$$G(x) = (1+x)^2(1+x^2)^3(1+x^4)^3 \cdots (1+x^{1024})^3.$$

We want to find the coefficients of x^{2025} and x^{2026} .

$$\begin{aligned} G(x) &= (1+x)^2(1+x^2)^3(1+x^4)^3 \cdots (1+x^{1024})^3 \\ &= \frac{1}{(1+x)(1-x)^3} \cdot [(1-x)(1+x)(1+x^2)(1+x^4) \cdots (1+x^{1024})]^3 \\ &= \frac{1}{(1-x^2)(1-x)^2} \cdot (1-x^{2048})^3 \\ &= (1+x^2+x^4+x^6+\cdots)(1+2x+3x^2+4x^3+\cdots)(1-3x^{2048}+3x^{4096}-x^{6144}). \end{aligned}$$

Since the degrees of the non-constant monomials in the last factor are 2048, 4096, and 6144, all of which are larger than 2026, the coefficients of x^{2025} and x^{2026} are the same as those in

$$(1+x^2+x^4+x^6+\cdots)(1+2x+3x^2+4x^3+\cdots).$$

To form x^{2025} , note that all powers in the first factor are even. Therefore, we only need to consider odd powers in the second factor. Hence the x^{2025} term in the product is

$$1 \cdot 2026x^{2025} + x^2 \cdot 2024x^{2023} + x^4 \cdot 2022x^{2021} + \cdots + x^{2024} \cdot 2x,$$

which gives

$$2026 + 2024 + \cdots + 2 = 1013 \cdot 1014 = 1027182.$$

Similarly, the coefficient of x^{2026} in $G(x)$ is

$$2027 + 2025 + \cdots + 1 = 1014^2 = 1028196.$$

Therefore, there are 1027182 ‘bad groups’ and 1028196 ‘good groups’.

(b) Consider the generating function for the total number of points:

$$G(x) = (1+x)^2(1+x^2)^4(1+x^4)^4 \cdots (1+x^{1024})^4.$$

We want to find the coefficients of x^{2025} and x^{2026} .

$$\begin{aligned} G(x) &= (1+x)^2(1+x^2)^4(1+x^4)^4 \cdots (1+x^{1024})^4 \\ &= \frac{1}{(1+x)^2(1-x)^4} \cdot [(1-x)(1+x)(1+x^2)(1+x^4) \cdots (1+x^{1024})]^4 \\ &= \frac{1}{(1-x^2)^2(1-x)^2} \cdot (1-x^{2048})^4 \\ &= (1+2x^2+3x^4+4x^6+\cdots)(1+2x+3x^2+4x^3+\cdots)(1-4x^{2048}+6x^{4096}-4x^{6144}+x^{8192}). \end{aligned}$$

By the same argument as in Question 1, the coefficients of x^{2025} and x^{2026} are determined by the first two factors. The coefficient of x^{2025} is

$$1 \cdot 2026 + 2 \cdot 2024 + 3 \cdot 2022 + \cdots + 1013 \cdot 2 = \sum_{n=1}^{1013} n(2028 - 2n).$$

Hence

$$\sum_{n=1}^{1013} n(2028 - 2n) = 2028 \cdot \frac{1013 \cdot 1014}{2} - 2 \cdot \frac{1013 \cdot 1014 \cdot 2027}{6} = \frac{1011 \cdot 1012 \cdot 2023}{6} = 347529910.$$

Similarly, the coefficient of x^{2026} is

$$1 \cdot 2027 + 2 \cdot 2025 + 3 \cdot 2023 + \cdots + 1014 \cdot 1 = \sum_{n=1}^{1014} n(2029 - 2n) = 348044515.$$

Therefore, there are 347529910 'bad groups' and 348044515 'good groups'.