MAA Metro NY Problem of the Month

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Let p and q be integers such that series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{109}-\frac{1}{110}+\frac{1}{111}=\frac{p}{q}$. Prove by elementary means that p is divisible by 167.

Computer or AI assisted/generated solutions will not be accepted.

Solution by Dr. Hari Kishan

Let p and q be integers such that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{110}+\frac{1}{111}=\frac{p}{q}. \qquad \ldots(1)$$

We have to show that p is divisible by 167.

The given series can be written as

$$\frac{p}{q}=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{111}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{110}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{111}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{55}\right)$$

$$=\frac{1}{56}+\frac{1}{57}+\cdots\frac{1}{110}+\frac{1}{111}$$

$$=\left(\frac{1}{56}+\frac{1}{111}\right)+\left(\frac{1}{57}+\frac{1}{110}\right)+\cdots\left(\frac{1}{83}+\frac{1}{84}\right) \qquad \text{(Rearranging the terms)}$$

$$=\frac{167}{56.111}+\frac{167}{57.110}+\cdots\frac{167}{83.84} \qquad \text{(Clearly167 and each denominator is relatively prime)}$$

$$=167\sum_{r=56}^{83}\frac{1}{r(167-r)}$$

$$=\frac{167m}{n}. \qquad \left(\text{Here } \sum_{r=56}^{83}\frac{1}{r(167-r)}=\frac{m}{n}\right)$$

Thus $p=167m$. This shows that p is multiple of 167. It means p is divisible by 167. Proved.