MAA Metro NY
November 2025, Problem of the Month

Prove by elementary means, using trigonometric identities, that

(a) $\cos(1^\circ)$ is irrational, and

(b) $\cos\left(\frac{1}{n}^\circ\right)$ is irrational, where n is a positive integer.

Computer or AI assisted/generated solutions will not be accepted.

Solution by Dr. Philip Cobb

Begin by proving by induction that there exist polynomials $f_n(x)$ and $g_n(x)$ with integer coefficients such that $\cos(nx) = f_n(\cos x)$ and $\sin(nx) = \sin x g_n(\cos x)$. If $n = 1$, then $f_1(x) = x$ and $g_1(x) = 1$.

Assume that both are true for $n = k$. Then

$$\begin{aligned}\cos(k+1)x = \cos(kx+x) = \cos(kx)\cos x - \sin(kx)\sin x &= f_k(\cos x)\cos x - g_k(\cos x)\sin^2(x) \\ &= f_k(\cos x)\cos x - g_k(\cos x)(1-\cos^2 x)\end{aligned}$$

so that $f_{k+1}(x) = f_k(x)x + g_k(x)(x^2-1)$. Similarly,

$$\sin(k+1)x = \sin(kx+x) = \sin(kx)\cos x + \cos(kx)\sin x = \sin x g_k(\cos x)\cos x + f_k(\cos x)\sin x$$

and $g_{k+1}(x) = g_k(x)x + f_k(x)$.

Now let $x = 1°$ and $n = 30$. Assume for a contradiction that $\cos 1°$ is a rational number r. Then $\cos(30x) = f_{30}(\cos x)$ and $\cos 30° = f_{30}(r)$ would be a rational number. But $\cos 30° = \sqrt{3}/2$ is irrational.

Finally, let $x = 1/n°$. Assume that $\cos 1/n°$ is rational. But $\cos(nx) = f_n(x)$ and $\cos(n \times 1/n°) = f_n(\cos 1/n°)$, showing that $\cos 1°$ is rational, a contradiction.