Write the infinite sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

As an infinite product of fractions of the form $\frac{n}{m}$, where n and m are different numbers in each fraction, but always satisfy $n = m + 1$.

SOLUTION.

It is known that the probability that two randomly chosen natural numbers are relatively prime is

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}. \qquad (1)$$

The solution to the famous **Basel Problem**, posed by Mengoli and solved by Euler, is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad (2)$$

Comparing (1) and (2), we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\frac{6}{\pi^2}} = \prod_{p \text{ prime}} \left(\frac{p^2}{p^2 - 1}\right),$$

a product of the desired form.

A good treatment of these topics for undergraduates can be found in Chapter 11 of *A Compact Capstone Course in Classical Calculus* by Peter R. Mercer (Birkhäuser, 2023).

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