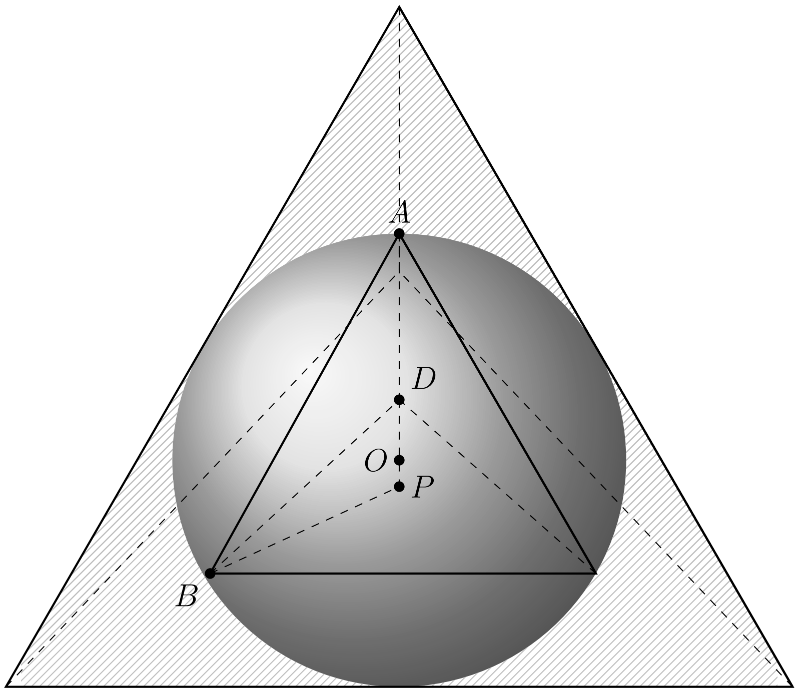
Solution by Dr. Asher Roberts

Consider the figure below.



Here O is the center of the unit tetrahedron and, also, the center of the circumscribing sphere. BPD is an isosceles triangle inside the equilateral triangle that is the bottom face of the unit tetrahedron. Since BPD is isosceles, each of its acute angles must be 30° (half of the angles of the equilateral triangle), so its obtuse angle is 120°. Thus we can use the law of cosines to find $|BP|$, i.e.,

$$\begin{aligned}|BD|^2 &= |BP|^2 + |DP|^2 - 2|BP||DP|\cos(120°)\\ 1 &= 2|BP|^2 - 2|BP|^2\left(-\frac{1}{2}\right),\end{aligned}$$

and so $|BP| = \frac{1}{\sqrt{3}}$. We can now use the Pythagorean theorem to find the height $|AP|$ of the unit tetrahedron by setting

$$\begin{aligned}|BP|^2 + |AP|^2 &= |AB|^2\\ \frac{1}{3} + |AP|^2 &= 1.\end{aligned}$$

Solving for $|AP|$ then gives us $|AP| = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$. The distance from the top of the tetrahedron to its center $|AO|$ is $\frac{3}{4}$ of that, and therefore the radius of the sphere is $\frac{\sqrt{6}}{4}$. Similarly, the radius of the sphere is $\frac{1}{4}$ of the height of the circumscribing tetrahedron, so the height of the circumscribing tetrahedron is $\sqrt{6}$. Similar to the way we calculated the height of the unit tetrahedron, we can work backward to determine that the side length of the circumscribed tetrahedron is 3. (The height is $\frac{\sqrt{6}}{3}$ times the side length, as we saw above.) The volume of the circumscribed tetrahedron is then

$$V_T = \frac{1}{3} \cdot (\text{base area}) \cdot (\text{height}) = \frac{1}{3}\left(\frac{\sqrt{3}}{4} \cdot 3^2\right) \cdot \sqrt{6} = \frac{9\sqrt{2}}{4},$$

while the volume of the sphere is

$$V_S = \frac{4}{3}\pi \cdot (\text{radius})^3 = \frac{4}{3}\pi\left(\frac{\sqrt{6}}{4}\right)^3 = \frac{\sqrt{6}}{8}\pi.$$

Hence the volume of the desired region is the difference between them, namely

$$\frac{9\sqrt{2}}{4} - \frac{\sqrt{6}}{8}\pi \approx 2.22.$$