MAA Metro NY
July 2025, Problem of the Month

Prove that a triangle with vertices p, q and r in the complex plane is equilateral if and only if $p^2 + q^2 + r^2 - pq - pr - qr = 0$.

Computer or AI assisted/generated solutions will not be accepted.

Solution: We provide a sketch of the solution. The sides of the triangle in question can be represented by the vectors $v_1 = p - q, v_2 = q - r$ and $v_3 = r - p$. Now $v_2 = e^{2\pi i/3} v_1$ and $v_3 = e^{4\pi i/3} v_1$ and since $e^{2\pi i/3}$ is a cube root of unity, we have that $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. It readily follows that $v_1^2 + v_2^2 + v_3^2 = v_1^2(1 + e^{2\pi i/3} + e^{4\pi i/3}) = 0$. We also have that $0 = v_1^2 + v_2^2 + v_3^2 = (p - q)^2 + (q - r)^2 + (r - p)^2$ which simplifies to $p^2 + q^2 + r^2 - pq - pr - qr = 0$. Steps that are easily reversible.