Show that it is impossible to construct an equilateral triangle in the plane with vertices that are lattice points.

SOLUTION.

Suppose we have an equilateral triangle in the plane with vertices that are lattice points. The area A of this simple polygon is given by Pick's Theorem as

$$A = I + \frac{B}{2} - 1,$$

where I is the number of lattice points interior to the triangle and B is the number of integer points on its boundary (including vertices and points along the sides). Since I and B are integers, A must be a rational number. However, the area of an equilateral triangle with side length s is given by the known formula $\frac{s^2\sqrt{3}}{4}$, which is irrational since s^2 is an integer. This contradiction establishes the non-existence of such a triangle.

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