We feature a solution submitted by Dr. John Chiarelli

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1 Question

Give explicit solutions for all positive integers n to the equation $a^2 + b^2 + c^2 = 77^n$ in positive integers.

2 Solution

Suppose that n is odd - i.e. that $n = 2k + 1$ for some non-negative integer k. Then, a possible solution is:

$$a = 4 * 77^k, b = 5 * 77^k, c = 6 * 77^k$$

We can verify this by noting that:

$$a^2 + b^2 + c^2 = (4 * 77^k)^2 + (5 * 77^k)^2 + (6 * 77^k)^2 = 4^2 77^{2k} + 5^2 77^{2k} + 6^2 77^{2k}$$

$$= 77^{2k}(4^2 + 5^2 + 6^2) = 77^{2k} * 77 = 77^{2k+1} = 77^n$$

Now, suppose that n is even - i.e. that $n = 2k + 2$ for some non-negative integer k. Then, one solution is:

$$a = 22 * 77^k, b = 33 * 77^k, c = 66 * 77^k$$

We can verify this by noting that:

$$a^2 + b^2 + c^2 = (22 * 77^k)^2 + (33 * 77^k)^2 + (66 * 77^k)^2 = 22^2 77^{2k} + 33^2 77^{2k} + 66^2 77^{2k}$$

$$= 77^{2k}(22^2 + 33^2 + 66^2) = 77^{2k} * 77^2 = 77^{2k+2} = 77^n$$

Since n must be odd or even, we have shown an explicit solution for all positive integers n.