MAA Metro NY

October/November 2024: Problem of the Month

Going the distance...

Show that the distance traveled by an underdamped spring-mass system with displacement $u(t) = 6e^{-t}\cos(t - \frac{\pi}{4})$ is of the form $a\sqrt{b}\coth\left(\frac{\pi}{c}\right)$, where a, b and c are integers. Computer or AI assisted/generated solutions will not be accepted.

A solution.

Rewrite $u(t) = 6e^{-t}\cos(t - \frac{\pi}{4}) = 3\sqrt{2}e^{-t}(\cos t + \sin t)$, by way of $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Taking the first derivative of $u(t)$, we get $u'(t) = -6\sqrt{2}e^{-t}\sin t$ and the distance S travelled by the system is computed from

$$S = \int_0^\infty |u'(t)|\ dt.$$

The horizontal intercepts of $u'(t)$ occur at $t = n\pi$, where $n = 0, 1, 2, ...$ In addition, $u'(t)$ alternate in sign from negative to positive on the intervals, $(0,\pi)$,$(\pi, 2\pi)$, $(2\pi, 3\pi)$, ... It follows that

$$\begin{aligned}
S &= \int_0^\pi |u'(t)|\ dt + \int_\pi^{2\pi} |u'(t)|\ dt + \int_{2\pi}^{3\pi} |u'(t)|\ dt + \cdots \\
&= -\int_0^\pi u'(t)\ dt + \int_\pi^{2\pi} u'(t)\ dt - \int_{2\pi}^{3\pi} u'(t)\ dt + \cdots \\
&= -(u(\pi) - u(0)) + (u(2\pi) - u(\pi)) - (u(3\pi) - u(2\pi)) + \cdots \\
&= u(0) - 2u(\pi) + 2u(2\pi) - 2u(3\pi) + \cdots \\
&= 3\sqrt{2} + \mathbf{6\sqrt{2}}\left(\mathbf{e^{-\pi} + e^{-2\pi} + e^{-3\pi}} + \cdots\right), \text{since } u(n\pi) = 3\sqrt{2}(-1)^n e^{-n\pi}, \text{for } n = 0, 1, 2, ...
\end{aligned}$$

The highlighted sum above is geometric with $|e^{-\pi}| < 1$ and we see that S equals

$$\begin{aligned}
S &= 3\sqrt{2} + \frac{6\sqrt{2}e^{-\pi}}{1 - e^{-\pi}} = 3\sqrt{2}\left(\frac{1 + e^{-\pi}}{1 - e^{-\pi}}\right) = 3\sqrt{2}\cdot\frac{e^{\pi/2}}{e^{\pi/2}}\left(\frac{1 + e^{-\pi}}{1 - e^{-\pi}}\right) = 3\sqrt{2}\left(\frac{e^{\pi/2} + e^{-\pi/2}}{e^{\pi/2} - e^{-\pi/2}}\right) \\
&= 3\sqrt{2}\coth\left(\frac{\pi}{2}\right).
\end{aligned}$$