One morning, in Springfield, somewhere in the US, it started snowing at a heavy but constant rate. Homer Simpson had just started his own snow-plow business. His snowplow started out at 8:00 A.M. At 9:00 A.M. it had gone 2 miles. By 10:00 A.M. it had gone 3 miles. Assuming that the snowplow removes a constant volume of snow per hour, determine the time at which it started snowing.

Solution by Henry Ricardo, Westchester Area Math Circle, New York.

Let t be the time measured in hours after 8 A.M. Let $x(t)$ denote the distance the snowplow has traveled at time t. (Note that $x(0) = 0, x(1) = 2$, and $x(2) = 3$.) Denote by $v(t)$ the volume of the snow at time t. Furthermore, let α be the constant rate of snow removal in cubic miles per hour and let k be the constant rate at which the snow falls (in cubic miles per hour). Finally, let h be the unknown number of hours before eight that it started snowing.. The change in volume is given by the rate at which the snow falls:

$$\frac{d}{dt}v(t) = k \quad \Rightarrow \quad v(t) = kt + c = k(t+h), \text{ where } v(-h) = 0, \; c = kh.$$

The rate α is proportional to the cross-section of the snow being plowed and the speed of the snowplow. If we assume the width of the plow is a constant w, then $\alpha = wv(t)\frac{d}{dt}x(t)$. Rearranging, we see that

$$\frac{d}{dt}x(t) = \frac{C}{t+h}, \quad C = \frac{\alpha}{kw}.$$

Integrating this separable ODE, we find that

$$x(t) = C \ln |t+h| + D.$$

The initial condition $x(0) = 0$ yields the constant of integration $D = -C \ln h$. Thus we have

$$x(t) = C \ln |t+h| - C \ln h = C \ln \left(1 + \frac{t}{h}\right). \tag{1}$$

Substituting the conditions $x(1) = 2, x(2) = 3$ in (1), we find that

$$\ln \left(1 + \frac{2}{h}\right) = \frac{3}{2} \ln \left(1 + \frac{1}{h}\right), \text{ or } \left(1 + \frac{2}{h}\right)^2 = \left(1 + \frac{1}{h}\right)^3.$$

Expanding and simplifying this last equation, we get $h^2 + h - 1 = 0$, with solution $(-1+\sqrt{5})/2 \approx 0.618$ hours (since h must be positive). Therefore the snow began about 37 minutes before eight—that is, at about $\boxed{7:23}$ A.M.