The Problem of the Month
June 2024

Find with justification all multisets of eight positive integers whose sum equals their product. For example, one such set is $\{1, 1, 1, 1, 1, 1, 2, 8\}$.

Computer or AI assisted/generated solutions will not be accepted.

We feature a solution submitted by Hannah Bahn, St Ann's School, Brooklyn NY.

MAA Problem of the Month June 2024

Hannah Bahn

June 2024

Find with justification all multisets of eight positive integers whose sum equals their product. For example, one such set is $\{1, 1, 1, 1, 1, 1, 2, 8\}$.

At a certain point, the product of the eight integers will begin to exceed their sum. This threshold occurs between the sets $\{1, 1, 1, 1, 1, 2, 2, 2\}$ and $\{1, 1, 1, 1, 2, 2, 2, 2\}$. Notice that the non-unit values in both sets are as small as possible, so that the growth of the product can be minimized. The first set has a sum of 11 and a product of 8. The second set has a sum of 12 and a product of 16. So, any set $\{1, 1, 1, 1, w, x, y, z\}$ will have a product that exceeds the sum; any set with even fewer ones will also have a product that is too large.

The set $\{1, 1, 1, 1, 1, 1, 1, z\}$ always yields a sum of $z+7$, while the product is z, so we can discount this set.

The sets that remain are $\{1, 1, 1, 1, 1, x, y, z\}$ and $\{1, 1, 1, 1, 1, 1, y, z\}$. The solution for the first type of set is the example given: $\{1, 1, 1, 1, 1, 1, 2, 8\}$. Through experiment, we can find the solution to the second type of set: $\{1, 1, 1, 1, 1, 2, 2, 3\}$.

More rigorously, we can prove that these two sets are the only solutions.

The question tells us that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 \cdot x_8. \quad (1)$$

So, assuming that $0 < x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8$, it readily follows that $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 \cdot x_8 \leq 8 \cdot x_8$.

We can divide both sides of the inequality by x_8 and get that $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 \leq 8$. We can now test different cases where $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 \leq 8$.

I.

$$
\begin{aligned}
x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 &= 1 \\
\text{or } \ x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 &= 1
\end{aligned}
$$

Now using (1),

$$7 + x_8 = x_8$$

Clearly this case leads to an empty set.

II.

$$
\begin{aligned}
x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 &= 2 \\
\text{or } \ x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1; x_7 &= 2
\end{aligned}
$$

Now using (1),

$$
\begin{aligned}
8 + x_8 &= 2 \cdot x_8 \\
\text{or } \ x_8 &= 8
\end{aligned}
$$

This case yields the solution $\{1, 1, 1, 1, 1, 1, 2, 8\}$.

III. If we use the same technique, this time setting $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 = 3$, it yields an empty set.

IV.

$$
\begin{aligned}
x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 &= 4 \\
\text{or } \ x_1 = x_2 = x_3 = x_4 = x_5 = 1; x_6 = x_7 &= 2
\end{aligned}
$$

Now using (1),

$$
\begin{aligned}
9 + x_8 &= 4 \cdot x_8 \\
\text{or } \ x_8 &= 3
\end{aligned}
$$

This case yields the solution $\{1, 1, 1, 1, 1, 2, 2, 3\}$.
Note: the case $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1; x_7 = 4$ leads to an empty set.

V. Continuing this technique, and looking at the cases:
$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \cdot x_7 = 5$ or 6 or 7 or 8.

We see that they lead to empty sets or duplicates of previously found sets. The only distinct solutions are {1, 1, 1, 1, 1, 1, 2, 8} and {1, 1, 1, 1, 1, 2, 2, 3}.