Problem of the Month - May 2024 Solution

Let $L_1(x)$ be the tangent line of a^x at $x = b$ and $L_2(x)$ be the tangent line of $\log_a x$ at $x = c$. For these two lines to be identical, we would need their slopes and y-intercepts to agree. Since

$$L_1(x) = a^b \ln a(x - b) + a^b$$

and

$$L_2(x) = \frac{1}{c \ln a}(x - c) + \log_a c,$$

if follows that

$$a^b \ln a = \frac{1}{c \ln a} \tag{1}$$

and

$$a^b - ba^b \ln a = \log_a c - \frac{1}{\ln a}. \tag{2}$$

From (1) it follows that

$$\begin{aligned} \log_a c &= -\log_a(a^b \ln^2 a) \\ &= -b - 2\log_a(\ln a). \end{aligned}$$

Combining the above result with (2), we have

$$a^b - ba^b \ln a + b + 2\log_a(\ln a) + \frac{1}{\ln a} = 0. \tag{3}$$

Let $h(x) = a^x - xa^x \ln a + x + K_a$ where $K_a = 2\log_a(\ln a) + \frac{1}{\ln a}$. Using the Intermediate Value Theorem, it is enough to find two values of x that make $h(x)$ positive and negative. We have $h(0) = 1 + K_a$ and $\lim_{x\to\infty} h(x) = -\infty$, so to finish the problem we must show that $1 + K_a > 0$. This follows from the monotonicity of the logarithms since $K_a > 2\log_a(\ln a) \geq 2\log_a(\ln e) = 0$.