The Problem of the Month

December 2023

Evaluate the integral below without the aid of AI or computer assistance. Give an exact answer.

$$\int_0^6 \frac{\sqrt{\ln(2023-x)}}{\sqrt{\ln(2023-x)}+\sqrt{\ln(x+2017)}}\,dx$$

SOLUTION by Henry Ricardo, Westchester Area Math Circle.

It is easy to see that the substitution $x \mapsto a+b-x$ yields $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$ Denoting our given integral by I and replacing x by $6-x,$ we have

$$I \;=\; \int_0^6 \frac{\sqrt{\ln(2017+x)}}{\sqrt{\ln(2017+x)}+\sqrt{\ln(2023-x)}}\,dx,$$

so that

$$\begin{aligned}
2I \;&=\; \int_0^6 \frac{\sqrt{\ln(2023-x)}}{\sqrt{\ln(2023-x)}+\sqrt{\ln(x+2017)}}\,dx \;+\; \int_0^6 \frac{\sqrt{\ln(2017+x)}}{\sqrt{\ln(2017+x)}+\sqrt{\ln(2023-x)}}\,dx \\
&= \int_0^6 \frac{\sqrt{\ln(2023-x)}\;+\;\sqrt{\ln(2017+x)}}{\sqrt{\ln(2023-x)}\;+\sqrt{\ln(2017+x)}}\,dx \\
&= \int_0^6 1\,dx \;=\; 6,
\end{aligned}$$

which implies that $\boxed{I=3}$.