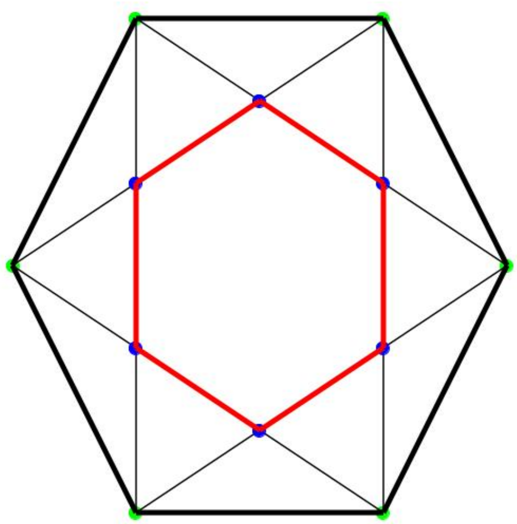
The Mathematics Department
Presents
The Problem of the Month
September 2023

The problem:

A Star of David is constructed by taking two identical equilateral triangles and superimposing one upon the other in such a way that the intersection of the triangles is a regular hexagon. Connecting each vertex of the star with its nearest neighbors by a line segment creates another, larger regular hexagon. (See diagram below.) Find the ratio of the area of the large hexagon to that of the smaller one.



The solution:

We begin by choosing a coordinate system. Start with a unit circle with center at the origin and place 6 equally spaced points on it: $[\cos\left(\frac{n\pi}{3}\right), \sin\left(\frac{n\pi}{3}\right)]\, n = 1 \ldots 6.$

Taking the first, third and fifth points as vertices generates one of the two triangles and doing the same with the second, fourth and sixth will generate the other. This will yield a Star of David. Note that using a unit circle as opposed to a circle with a different radius is of no consequence as scaling up or down will affect the areas of the two hexagons equally, leaving the ratio of their areas unchanged.

Now note that we can derive the area of a regular hexagon simply by knowing the length of one of the 6 equal sides.

If the length of a side is x, then the area of the hexagon will be $\frac{3\sqrt{3}x^2}{2}$. The reason for this is that a regular hexagon with side x is made up of 6 equilateral triangles with sides all equal to x. Each of these triangles will have area $\frac{\sqrt{3}x^2}{4}$. Multiplying by 6 gives the result.

Now our construction makes clear that the length of a side of the large hexagon is 1, yielding an area of $\frac{3\sqrt{3}}{2}$. For the smaller hexagon we find each edge to be of length $\frac{\sqrt{3}}{3}$. This makes the area of the small hexagon to be $\frac{\sqrt{3}}{2}$.

Thus the ratio we seek is 3.