The Problem of the Month Solution
June 2023

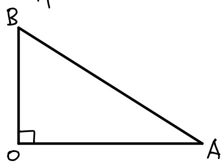
In a 3-4-5 right triangle, if a curve γ joins points of its two legs to bisect the area of the triangle, identify and find the length of the shortest possible such curve.

Solution by Teawoo Kim, St. Paul's School, Concord, NH, USA

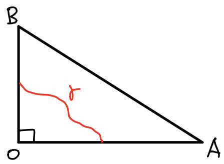
Claim: γ is a quarter circle with radius $\sqrt{\frac{12}{\pi}}$ that is centered at the vertex of the right angle. The length of γ is $\sqrt{3\pi}$.

Proof:

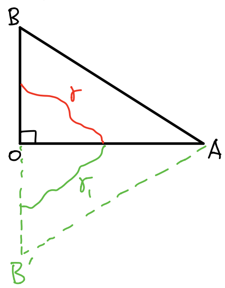
Suppose there is a right triangle $\triangle OAB$ shown as below:

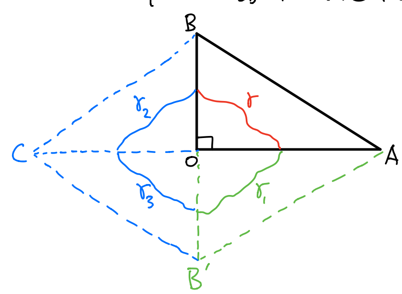


We can draw the curve γ as:



Reflect the figure with respect to $\overline{OA}$ to make a triangle $\triangle BB'A$, then reflect $\triangle BB'A$ with respect to $\overline{BB'}$ to make the figure:





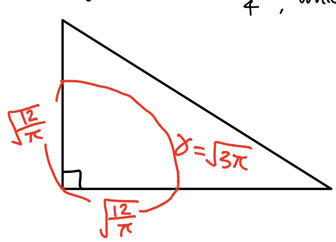
Now we obtain a closed curve γ-γ_1-γ_2-γ_3, having the area of 12.

From the isoperimetric inequality, of all the areas that take up the same space, circle has the minimum perimeter.

Thus, the closed curve γ-γ_1-γ_2-γ_3 should be a circle.

The circle with area 12 has a radius $\sqrt{\frac{12}{\pi}}$, and the perimeter is $2\pi\sqrt{\frac{12}{\pi}}$.

The length of γ is $\frac{2\pi\sqrt{\frac{12}{\pi}}}{4}$, which is $\sqrt{3\pi} \approx 3.06998$.



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