Solutions to the Problem of the Month
May 2023

Find the exact value of

$$\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{7}\right).$$

Give your answer as a rational multiple of π. Computer or AI assisted/generated solutions will not be accepted.

Solution I.

Let $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ so that $\tan\theta = \frac{1}{2}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(\frac{1}{2})}{1-(\frac{1}{2})^2} = \frac{4}{3}$. Also, letting $\gamma = \tan^{-1}\left(\frac{1}{7}\right)$ results in $\tan\gamma = \frac{1}{7}$.

Remark: $0 < \tan^{-1}\left(\frac{1}{7}\right) < \tan^{-1}\left(\frac{1}{2}\right) < \tan^{-1}(1) = \frac{\pi}{4}$ or equivalently $0 < \gamma < \theta < \frac{\pi}{4}$, so that $0 < 2\theta - \gamma < \frac{\pi}{2}$.

Now, $\tan(2\theta - \gamma) = \frac{\tan 2\theta - \tan\gamma}{1+\tan 2\theta \tan\gamma} = \frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\cdot\frac{1}{7}} = 1$. It follows that $2\theta - \gamma = \frac{\pi}{4}$, or $\theta - \frac{1}{2}\gamma = \frac{\pi}{8}$, and we see that $\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{8}$.

Solution II.

By using the complex argument, we see that

$$\begin{aligned} 2\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{7}\right) &= 2\arg(2+i) - \arg(7+i) = \arg(2+i)^2 - \arg(7+i) \\ &= \arg(3+4i) - \arg(7+i) = \arg\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z}. \end{aligned}$$

By the remark above, $2\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$, and dividing by 2, gives us the value of the expression in question, namely $\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{8}$.