The Problem of the Month, February 2023 solutions.

Several solvers provided multiple solutions. We feature three solutions provided by Dr. Henry Ricardo.

For a right triangle with legs of length a and b and hypotenuse c, show that $a+b \le c\sqrt{2}$. Classify all the right triangles where equality occurs.

Solution 1 by Henry Ricardo, Westchester Area Math Circle

The Pythagorean Theorem gives us $a^2+b^2=c^2$. Then the Quadratic Mean inequality yields

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}, \text{ or } a+b \le \frac{2}{\sqrt{2}}\cdot\sqrt{a^2+b^2} = c\sqrt{2},$$

with equality if and only if $a=b$—that is, if and only if the right triangle is *isosceles*.

Solution 2 by Henry Ricardo, Westchester Area Math Circle

The function $f(x)=x^2$ is convex, so that we may use Jensen's inequality to get

$$\left(\frac{a+b}{2}\right)^2 \le \frac{a^2+b^2}{2} = \frac{c^2}{2}, \quad \text{or} \quad a+b \le c\sqrt{2},$$

with equality if and only if $a=b$—that is, if and only if the right triangle is *isosceles*.

Solution 3 by Henry Ricardo, Westchester Area Math Circle

The Cauchy-Schwarz inequality gives us

$$\left(\frac{a}{2}\cdot 1+\frac{b}{2}\cdot 1\right)^2 \le \left(\frac{a^2}{4}+\frac{b^2}{4}\right)\left(1^2+1^2\right), \text{ or (after some simple algebra) } a+b \le c\sqrt{2}.$$

Equality holds if and only if $a = b$—that is, if and only if the triangle is *isosceles*.