Problem of the Month December 2022

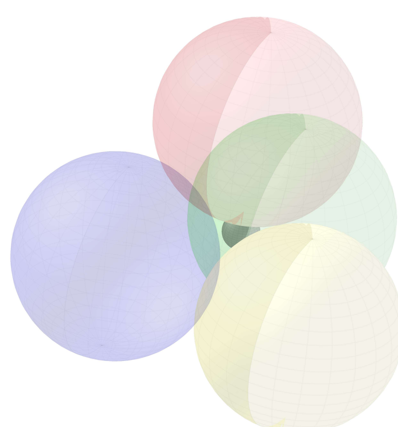
Take four unit spheres and stack them like cannon balls. Find the radius of the largest sphere that can be placed inside the cavity inside the cannon ball pyramid.

```
> restart :
> with(plots) :
> with(plottools) :
> a := sphere([0,-1, 0], 1, transparency = .9, color = blue) :
> b := sphere([0, 1, 0], 1, transparency = .9, color = yellow) :
> c := sphere([-sqrt(3), 0, 0], 1, transparency = .9, color = green) :
```

> $dd := sphere\left(\left[-\frac{1}{\text{sqrt}(3)}, 0, \frac{2\cdot\text{sqrt}(2)}{\text{sqrt}(3)}\right], 1, transparency = .9, color = red\right)$:

> $e := sphere\left(\left[-\frac{1}{\text{sqrt}(3)}, 0, \frac{1}{\text{sqrt}(6)}\right], \frac{\text{sqrt}(2)}{\text{sqrt}(3)} - 1, transparency = .8, color = black\right)$:

```
> display([a, b, c, dd, e], scaling = constrained, axes = none);
```



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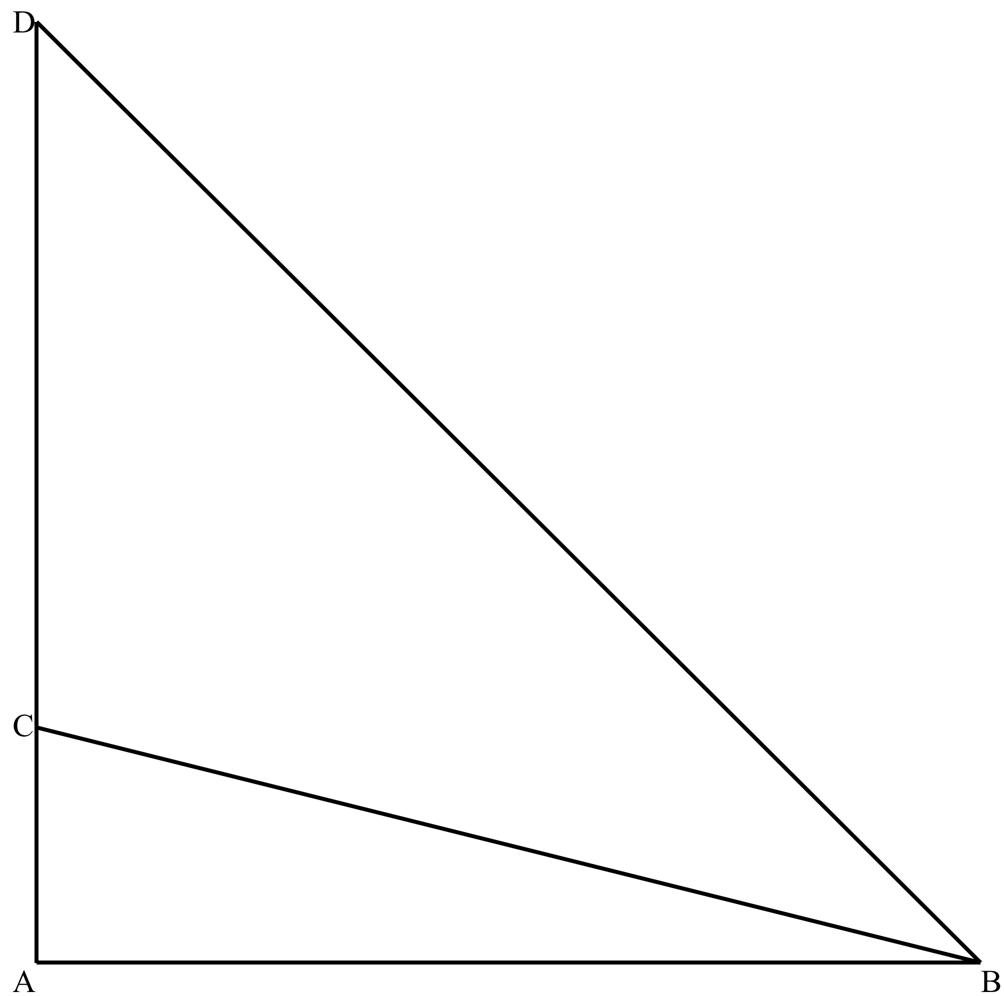
Solution: Place three unit spheres on a table. Their centers form an equilateral triangle with each side 2 units long. The altitude of this triangle is $\sqrt{3}$. The incenter of this equilateral triangle (the center of the inscribed circle) is the point where the angle bisectors meet. This point is $\frac{2}{\sqrt{3}}$ units from the center of each of the base spheres. Simple trigonometry of the 30-60-90 triangle makes this clear. The center of the apex unit sphere in our cannon ball stacking, as well as the small sphere that just fits into the cavity, will lie directly above this point, the incenter.

Now, consider the following diagram:

```
> restart:
> with(plots):
> with(plottools):
> a := line([0, 0], [2/sqrt(3), 0]):
> b := line([0, 0], [0, 2*sqrt(2)/sqrt(3)]):
> c := line([2/sqrt(3), 0], [0, 1/sqrt(6)]):
> dd := textplot([0, 0, "A"], align = {left, below}): e := textplot([2/sqrt(3), 0, "B"], align
      = {right, below}):
> f := textplot([0, 1/sqrt(6), "C"], align = {left}):
> g := textplot([0, 2*sqrt(2)/sqrt(3), "D"], align = {left}): h := line([2/sqrt(3), 0], [0,
      2*sqrt(2)/sqrt(3)]):
> display([a, b, c, dd, e, f, g, h], axes = none);
```



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Here A is the incenter of the equilateral triangle made up of the centers of the 3 base spheres. B is the center of one of the base spheres. It doesn't matter which one. D is the center of the apex unit sphere, the one that sits at the top of the stacking. Now the length of AB is $\frac{2}{\sqrt{3}}$ by reason of the above discussion. The length of BD is 2, as it is the distance between the centers of two tangent unit spheres. The Pythagorean theorem tells us that the length of AD is $2 \cdot \frac{\sqrt{2}}{\sqrt{3}}$.

Now, let C be the center of the mystery sphere, the one whose radius we seek. Let *x* be its radius. This tells us that the length of CD and of BC is $1 + x$. It also tells us that the length of AC is $2 \cdot \frac{\sqrt{2}}{\sqrt{3}} - (1 + x)$.

Now, we know the lengths of AB, BC and AC, in terms of *x*. Applying the Pythagorean theorem and solving for *x* yields:

$$x = \frac{\sqrt{3}}{\sqrt{2}} - 1.$$

This is the radius of the small sphere that just fits into the cavity made by the cannon ball stacking of the 4 unit spheres.