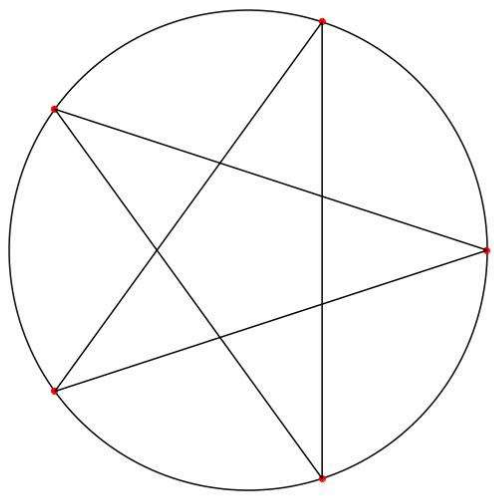
The Problem of the Month
September 2022

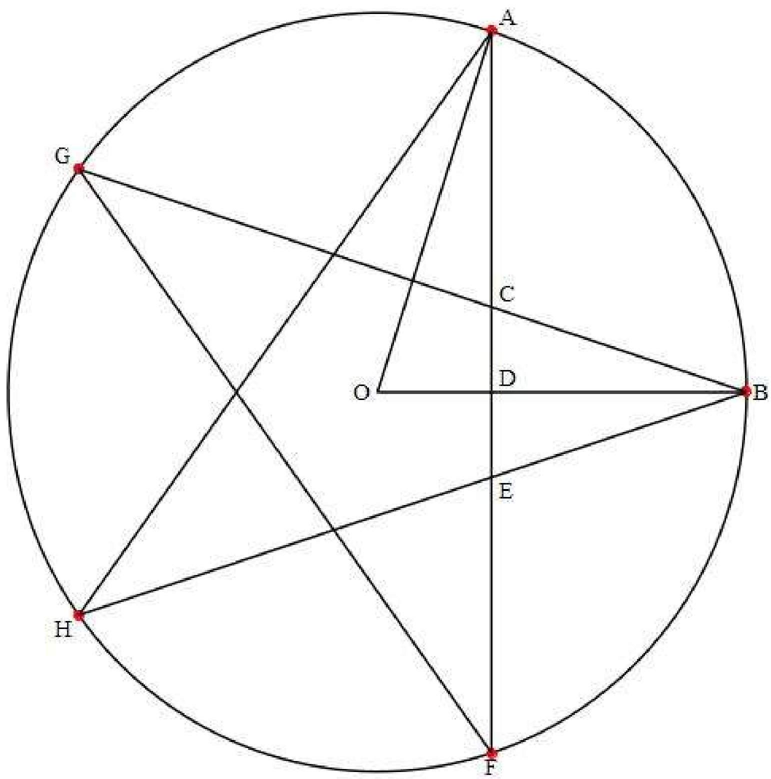
The Problem:

Consider a regular five-pointed star inscribed in a circle of radius 1. Find the area inside the circle but outside the star.



The Solution:

Consider the following diagram:



Here O is the center of the circle. We have drawn lines from O to two vertices of the star, A and B. Points C and E are where BG and BH intersect AF, respectively, and D is where OB intersects AF. Note that triangle CEB is a ***golden triangle***, an isosceles triangle with apex angle 36 and base angles 72 degrees. Let us call the lengths of line segments |CB| and |CE| *a* and *b*, respectively. Then, because triangle CEB is golden, we have:

$$\frac{a}{b} = \Phi = \frac{1+\sqrt{5}}{2}$$

This allows us to relate the base and the height of triangle CEB as follows:

$$h = \sqrt{(b\Phi)^2 - \left(\frac{b}{2}\right)^2} = b\sqrt{\Phi^2 - \frac{1}{4}} = \frac{b}{2}\sqrt{5 + 2\sqrt{5}}$$

Now, observe that triangle AOD is a right triangle (right angle at D) with |OA| = 1 and angle AOD being 72 degrees. Thus |OD| = cos(72) ≈ 0.309. This, in turn makes |DB| = 1 – 0.309 = 0.691 = *h*. Using the relationship between *b* and *h* that we derived above, we get:

$$b = \frac{1.382}{\sqrt{5+2\sqrt{5}}}$$

Now that we have values for *b* and *h*, we find the area of each of the golden triangles that make up the points of the star to be:

$$\frac{0.691}{\sqrt{5+2\sqrt{5}}} \times 0.691 \approx 0.1551$$

Multiplying by 5 gives the total area of the 5 golden triangles to be 0.7757.

This leaves the central pentagon. It consists of 5 isosceles triangles with apex angle = 54 degrees and a base equal to $b = \frac{1.382}{\sqrt{5+2\sqrt{5}}}$. The altitude of each of these 5 triangles will be $H = \frac{b}{2}\tan(54)$.

This gives the area of the pentagon to be $5(\frac{b}{2})^2\tan(54)$ ≈ 0.3469.

Putting this all together, we see that the area enclosed by the unit circle but outside the inscribed star to be:

π – (0.7757 +0.3469) = π – (1.1226) = 2.019