The Problem of the Month
June 2022

Find the volume of the solid obtained by rotating the region enclosed by the curves

$$y = \frac{1}{1+x^4}, \quad y = 0, \quad x \geq 0$$

about the x-axis. Compute any resulting definite integral without the aid of a computer†. Give your answer in the form $\frac{a\pi^2\sqrt{b}}{c}$, where a, b and c are positive integers.

A solution. The volume of the resulting solid by way of the disk method can be found by evaluating the definite integral

$$\pi \int_0^\infty \left(\frac{1}{1+x^4}\right)^2 dx = \pi \int_0^\infty \frac{1}{(1+x^4)^2}\, dx.$$

We will evaluate the integral $\int_0^\infty \frac{1}{(1+x^4)^2}\, dx$ with the help of the integration by parts technique.

We begin with an application of the integration by parts formula, $\int u dv = uv - \int v du$, to the definite integral $\int_0^\infty \frac{1}{1+x^4}\, dx$. Let $u = (1+x^4)^{-1}$ and $dv = dx$ from which it readily follows that $du = -4x^3(1+x^4)^{-2}dx$ and $v = x$. Now,

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^4}\, dx &= \left[\frac{x}{1+x^4}\right]_0^\infty + 4\int_0^\infty \frac{x^4}{(1+x^4)^2}\, dx \\
&= 4\int_0^\infty \frac{x^4}{(1+x^4)^2}\, dx \\
&= 4\int_0^\infty \frac{1+x^4}{(1+x^4)^2}\, dx - 4\int_0^\infty \frac{1}{(1+x^4)^2}\, dx \\
&= 4\int_0^\infty \frac{1}{1+x^4}\, dx - 4\int_0^\infty \frac{1}{(1+x^4)^2}\, dx.
\end{aligned}$$

Rearranging the terms above and applying † give us

$$\int_0^\infty \frac{1}{(1+x^4)^2}\,dx = \frac{3}{4}\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{3\pi\sqrt{2}}{16}.$$

The volume is now seen to be

$$\pi\int_0^\infty \frac{1}{(1+x^4)^2}\,dx = \frac{3\pi^2\sqrt{2}}{16}.$$

† You may assume that $\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi\sqrt{2}}{4}$. See for example, Yusuf Z. Gürtaş (2022), An Unorthodox Approach to Skinning a Definite Integral, The College Mathematics Journal, 53:2, 134-139, DOI: 10.1080/07468342.2022.2011543