The Problem of the Month
May 2022

Find the area of the region enclosed by the curve $x^4 + y^4 = x^2 + y^2$. Compute any resulting definite integral without the aid of a computer. Give your answer in the form $\pi\sqrt{a}$, where a is a positive integer.

A solution.

The curve is symmetric about the x and y axes and the lines $y = \pm x$. Converting to polar form, with $x = r\cos\theta$ and $y = r\sin\theta$, readily gives $r^2 = \frac{1}{\cos^4\theta + \sin^4\theta}$. Using the symmetries of the curve, we can write the area of the region as

$$8 \cdot \frac{1}{2} \int_0^{\pi/4} r^2 \, d\theta.$$

An application of the identities, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, gives us $r^2 = \frac{2}{\cos^2 2\theta + 1} = \frac{2\sec^2 2\theta}{1+\sec^2 2\theta} = \frac{2\sec^2 2\theta}{2+\tan^2 2\theta}$, and the area can be rewritten as

$$4 \int_0^{\pi/4} \frac{2\sec^2 2\theta}{2 + \tan^2 2\theta} \, d\theta.$$

To compute the above integral, let $t = \tan 2\theta$. It follows that $dt = 2\sec^2 2\theta \, d\theta$ and we get

$$4 \int_0^{\infty} \frac{1}{2+t^2} \, dt = \frac{4}{\sqrt{2}} \left[\lim_{t\to\infty} \arctan\left(\frac{t}{\sqrt{2}}\right) - \arctan(0) \right] = \pi\sqrt{2}.$$