Find the area of the region enclosed by the curve $x^4 + y^4 = x^2 + y^2$. Compute any resulting definite integral without the aid of a computer. Give your answer in the form $\pi \sqrt{a}$, where $a$ is a positive integer.

A solution.

The curve is symmetric about the $x$ and $y$ axes and the lines $y = \pm x$. Converting to polar form, with $x = r \cos \theta$ and $y = r \sin \theta$, readily gives $r^2 = \frac{1}{\cos^4 \theta + \sin^4 \theta}$. Using the symmetries of the curve, we can write the area of the region as

$$8 \cdot \frac{1}{2} \int_0^{\pi/4} r^2 \, d\theta.$$

An application of the identities, $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ and $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$, gives us $r^2 = \frac{2}{\cos^2 2\theta + 1} = \frac{2 \sec^2 2\theta}{1 + \tan^2 2\theta}$, and the area can be rewritten as

$$4 \int_0^{\pi/4} \frac{2 \sec^2 2\theta}{2 + \tan^2 2\theta} \, d\theta.$$

To compute the above integral, let $t = \tan 2\theta$. It follows that $dt = 2 \sec^2 2\theta \, d\theta$ and we get

$$4 \int_0^{\infty} \frac{1}{2 + t^2} \, dt = \frac{4}{\sqrt{2}} \left[ \lim_{t \to \infty} \arctan \left( \frac{t}{\sqrt{2}} \right) - \arctan (0) \right] = \pi \sqrt{2}.$$