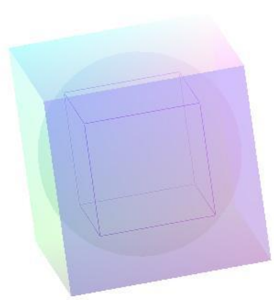
The Problem of the Month
August 2021

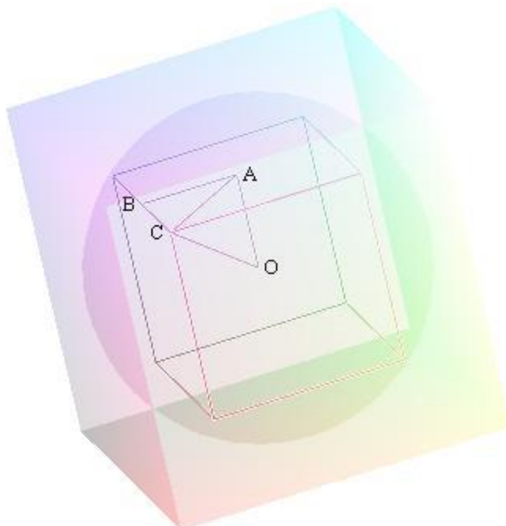
The Problem:

Imagine a cube with each edge being of length 1, i.e. a unit cube. Circumscribe this cube with a sphere. That is to say, draw the smallest sphere that contains this cube. Now, circumscribe the sphere with a cube. In other words, place the sphere (with the unit cube inside) inside the smallest cubic box that could contain it. (See the picture below.) Find the volume of the region contained inside the larger circumscribing cube that is ***not*** contained inside the sphere. Put another way, if the sphere were solid, how much air would there be inside the circumscribing box?



The Solution:

Consider the picture, below:



Here "O" is the center of the unite cube and, also, the center of the circumscribing sphere. The point "A" is the center of the top face of the cube and the point "B" is the midpoint of one edge of the cube. Lastly, point "C" is a corner of the top face of the cube.

Clearly |BC| = 1/2 and |AB| = 1/2. By the Pythagorean Theorem, this makes $|AC| = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

Now since |OA| = 1/2, we have $|OC| = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

This makes the radius of the circumscribing sphere equal to $\frac{\sqrt{3}}{2}$.

From this, we conclude that each edge of the larger circumscribing cube has length $2\frac{\sqrt{3}}{2} = \sqrt{3}$ and its volume is $3\sqrt{3}$.

To find the volume of the region contained within the large cube that is not inside the sphere, we simply subtract the volume of the sphere from the volume of the cube.

The formula for the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

Substituting our value for the radius of the sphere we get:

$$V = \frac{4}{3}\pi\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{4}{3}\pi\left(\frac{3\sqrt{3}}{8}\right) = \pi\left(\frac{\sqrt{3}}{2}\right).$$

Finally, we subtract the volume of the sphere from the volume of the cube to get:

$$\sqrt{3}\left(3 - \frac{\pi}{2}\right) \approx 2.47$$