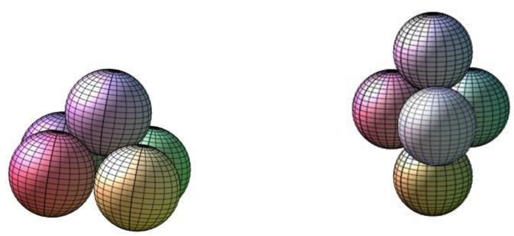
<u>The Problem of the Month</u>
<u>July, 2021</u>

The Problem:

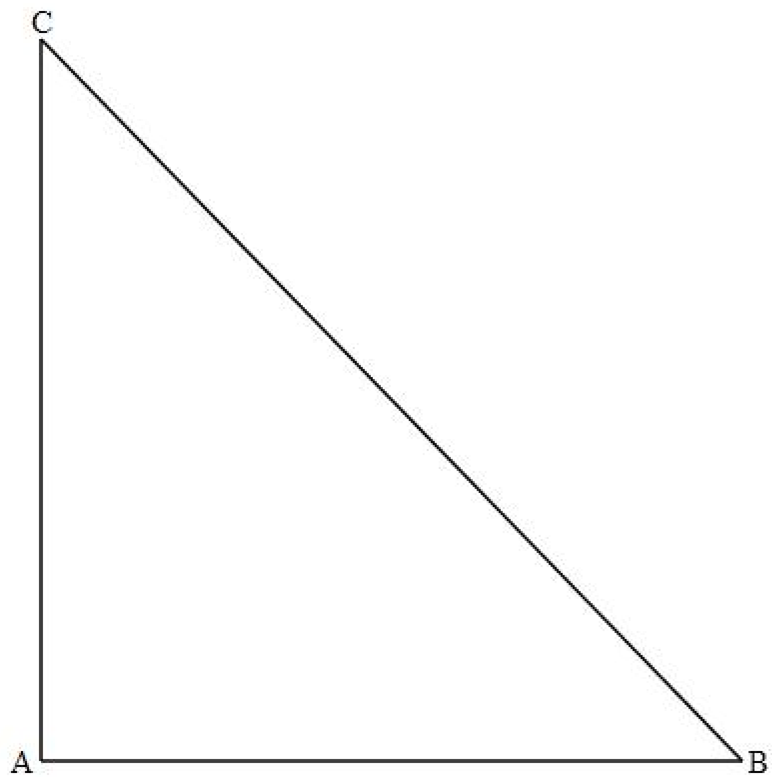
You work in the shipping department of Mister Adderley's Cannonball Emporium. A customer has just ordered five cannonballs. Your job is to pack them in the smallest box, by volume. The box must have rectangular parallel opposite sides. You consider two configurations. The first places four of the balls with their centers making a square. The fifth ball is then placed in the central indentation on top. See below, left. The second configuration places three balls so that their centers make an equilateral triangle and the two remaining balls are placed in the central indentations above and below the three making up the triangle. See below, right. Find the length, width and height the smallest box in each case, given that each cannonball has radius 1.



The Solution:

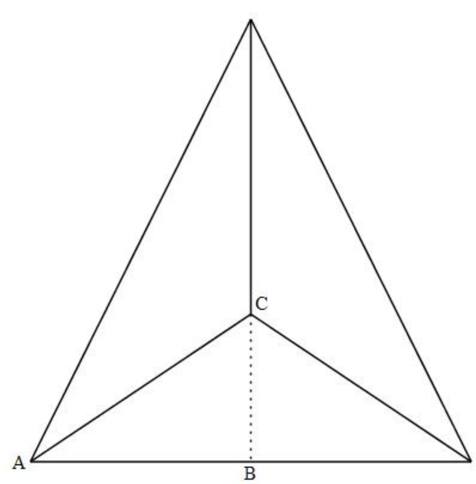
Let us consider the first configuration. Imagine that the four balls making up the square are placed on a flat table. The center of the square made by the centers of the four balls will be $\sqrt{2}$ from each center.

Consider the right triangle below. Here A is the center of the square, B is the center of one of the four cannon balls and C is the center of the fifth.



As $|AB| = \sqrt{2}$ and $|BC| = 2$, we conclude that $|AC| = \sqrt{2}$. Thus the center of the fifth ball is $1 + \sqrt{2}$ above the xy-plane.

Now let us consider the second configuration. The center of the fourth ball sitting in the indentation on top of the triangle and center of the fifth ball sitting in the indentation below the triangle will be above and below the in-center of the equilateral triangle with each side being of length 2. Consider the following diagram:



Here, we have |AB| = 1 and angle BAC = π/6. This yields $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}=\frac{1}{|AC|}\Rightarrow |AC|=\frac{2}{\sqrt{3}}\Rightarrow |BC|=\frac{1}{\sqrt{3}}$.

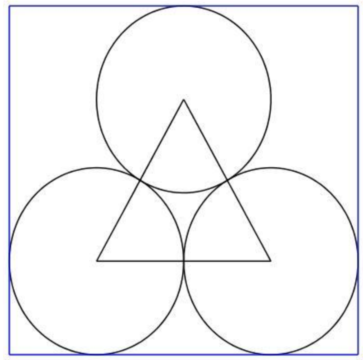
Now let us determine the height of the center of the fourth cannon ball (the one on top) above the plane determined by the centers of the three that form the equilateral triangle. Let us use the same right triangle ABC above but this time |AB| = $\frac{2}{\sqrt{3}}$. This the distance from the center of one of the balls that make up the triangle to the in-center of the triangle as we have just determined. We also have |BC| = 2 as this is the distance from the center of one of the three balls in the triangle and the center of the top ball. This yields $|AC|=\frac{2\sqrt{2}}{\sqrt{3}}$.

Putting this together, we see that the center of the fourth (top) ball will be $1+\frac{2\sqrt{2}}{\sqrt{3}}$ above the xy-plane and the center of the fifth (bottom) ball will be $\frac{2\sqrt{2}}{\sqrt{3}}-1$ below the xy-plane.

Now for the enclosing boxes. For the first configuration clearly the length and width of the box will both be 4. The height will be 1 unit higher than the center of the fifth ball so the height will be $2+\sqrt{2}$. The volume of this box will be:

4 x 4 x ($2+\sqrt{2}$) = 32 + 16$\sqrt{2}$ ≈ 54.6 = the volume of the smallest box for the first configuration.

For the second configuration, the width will be 4 and the length will be $2+\sqrt{3}$ the altitude of the equilateral triangle plus 2 (one for the radii at the base and one for the radius at the apex). We see this in the following diagram:



The height of the box will be determined by the centers of the upper and lower balls (numbers 4 and 5). The top of the top ball will be $2+\frac{2\sqrt{2}}{\sqrt{3}}$ above the xy-plane. The bottom of the bottom ball will be $\frac{2\sqrt{2}}{\sqrt{3}}$ below the xy-plane. Thus the total height of the box will be $2+\frac{4\sqrt{2}}{\sqrt{3}}$.

This makes the volume for the smallest box for the second configuration approximately 78.6.