The Problem of the Month
June 2021

The Problem:

Take a standard deck of 52 playing cards and place them on the table, face down. Let us say that the top card is in position 1 and the bottom card is in position 52 and the other card positions are numbered accordingly. Cut the deck precisely in two and take the portion that was the top 26 cards in your right hand and the bottom portion in your left. Execute a perfect riffle shuffle interlacing the cards from each hand such that the bottom card in your left hand goes on the bottom followed by the bottom card in your right hand, and so on. In this way, when you are done, the bottom card of the deck will remain on the bottom and the top card will remain on the top. This is called a perfect out-shuffle. (Surprisingly, executing 8 perfect out-shuffles in a row will return each card in the deck to its initial position!) Find all positions in the deck, other than 1 and 52, that return to their initial spot after executing two perfect out-shuffles.

The Solution:

First observe that the function $f(n)$ that gives the position of the nth card after one shuffle is a piecewise function. For cards in the top 26, the function is given by $f(n) = 2n - 1$. This should be clear as the cards, 1 through 26 are interlaced with cards from the bottom half of the deck. For the cards in the bottom half, the function is given by $f(n) = 2(n - 26)$. This, too, should be clear. The value $n - 26$ tells us how close to the top of the left hand part of the deck the card is. This value gets doubled as there are as many cards from the right hand part that go above it. For example, when $n = 27$, we are on top of the left hand pack. It gets only one card from the right hand pack above it, so it ends up in spot 2. When $n = 28$, the card in spot 27 and two cards from the right hand pack go above it. So it ends up in spot 4.

Only cards initially in spots 1, 18, 35 and 52 return to their initial spots after two perfect riffle out-shuffles. Cards in spots 1 and 52 remain fixed after any number of shuffles. This is immediate from the formulation of the function. Cards in spot 18 & 35 alternate with one another. After 2 shuffles, cards 1, 18, 35 and 52 remain the only cards in their original locations.

Here's why. Recall the rule: The cards in your right hand will follow the rule $n \rightarrow 2n-1$ and the cards in your left hand will follow the rule $n \rightarrow 2(n-26)$. First observe that, for cards in your right hand, $f(n) \geq n$ and for cards in your left hand, $f(n) \leq n$. Thus, if a card is in your right hand, say spot k, and applying the function once leaves it in your right hand then $f(f(k))$ will be greater than k. To move the card from the right hand to the left hand, k must be greater than or equal to 14. Similarly, if a card is in spot k in your left hand, and applying the function leaves it in your left hand, then $f(f(k))$ will be smaller than k. Thus, in order to ensure that the card gets transferred to your right hand, k must be less than or equal to 39. Thus, if $f(f(k))$ is to equal k after two shuffles, then the card needs to have been transferred back and forth between the two hands.

This means that if the card in spot k started off in your right hand, then k must be even since it returns from your left hand by the left hand rule, $n \rightarrow 2(n-26)$.

Now, let us find k such that it returns to spot k after two applications of f. Start with k in your right hand. The number k must be greater than or equal to 14 to ensure that $f(k)$ is greater than or equal to 27. Apply f twice and set the result equal to k.

2*((2*k - 1) - 26) = k implies 2*k - 27 = k/2 (recall k is even).

Now 2*k - k/2 = 27 implies 4*k - k = 54 which implies 3*k = 54 which yields k = 18. Thus, the only card in your right hand (other than 1) that returns to its initial position after two shuffles is in spot 18.

Alternatively, begin with k in your left hand and less than or equal to 39 and repeat the argument above:

Now $2*(2*k - 52) - 1 = k$ implies $(4*k - 104) - 1 = k$ which implies $4*k - k = 105$ which implies $3*k = 105$. This yields $k = 35$. Thus, the only card in your left hand (other than 52) that returns to its initial position after two shuffles is in spot 35.

Thus $f(18) = 35$ & $f(35) = 18$ so $f(f(18))= 18$ & $f(f(35)) = 35$.
Moreover 18 & 35 are the only two such values.