Problem of the Month for April 2021

Find the volume of the largest right circular cone that can be inscribed in a unit sphere.

<u>The Solution</u>:

Start with the cone inscribed in the sphere with its apex at the north pole. If the base of the cone is in the northern hemisphere, it is obvious that the volume will increase as the base descends towards the equator since both the height of the cone and the radius of the base are increasing. When the base reaches the equator, both the height and the radius will equal 1 and we get a volume of:

$$V = \frac{\pi}{3} \approx 1.047$$

If we continue to allow the base to descend, the height of the cone will continue to increase but the radius will begin to shrink. Let us check to see if the volume continues to increase or if it begins to shrink.

Let us take the cross section of the sphere in the *yz*-plane. Let *h* be the height of our cone and let *r* be the radius of the base.

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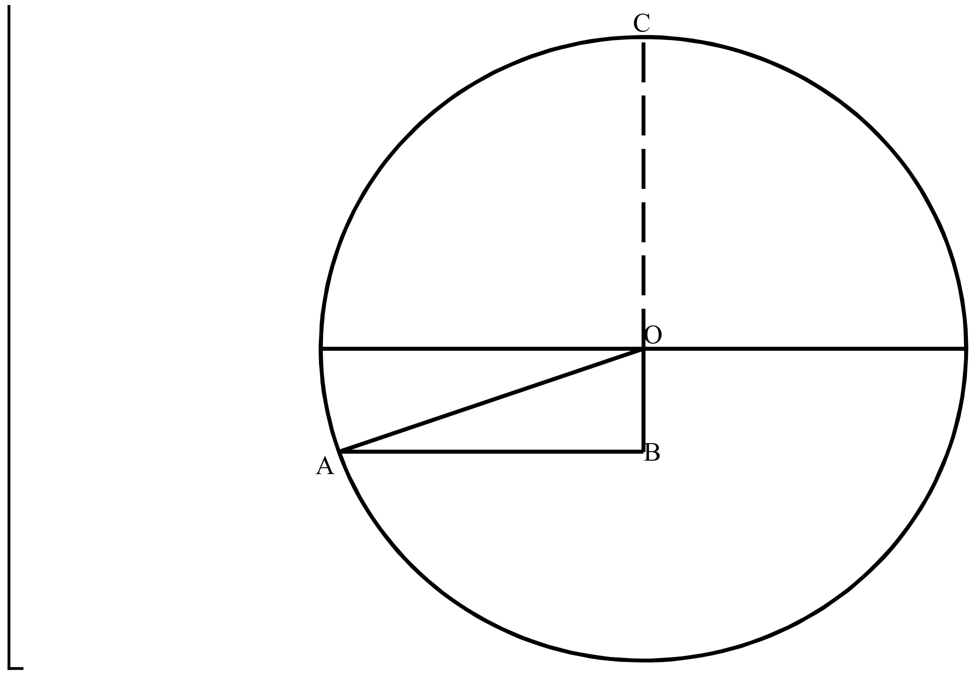

In the diagram above, |OC| = 1, |BC| = *h*, |AB| = *r* and |OB| = *h*-1.

Now *h* and *r* are related by $r^2 + (h-1)^2 = 1$. This gives $r = \sqrt{2h - h^2}$.

Now, the volume of a cone is 1/3 times the area of the base times the height. Thus:

$$V = \frac{1}{3}\left(\pi r^2\right) \times h = \frac{\pi}{3}\left(h\left(2h - h^2\right)\right) = \frac{\pi}{3}\left(2h^2 - h^3\right)$$

Let us take a value of *h* that places the base just below the equator and see what happens to the volume. Plugging $h = 1.1$ into the formula for *V* gives approximately 1.14. This is larger than the volume when the base is at the equator. Thus the volume is continuing to grow as we descend. Clearly this cannot continue since we will get a value of zero for the volume when we come to the south pole. Thus, there must be a value for *h* between 1 and 2 where the volume is maximal.

To maximize this area we differentiate V with respect to *h* and set this equal to zero.

$$\frac{dV}{dh} = \frac{\pi}{3}\left(4\,h - 3\,h^2\right)$$

Setting this equal to 0 yields

$$4\,h - 3\,h^2 = 0 \Rightarrow h(4 - 3\,h) = 0 \;\Rightarrow 4 - 3\,h = 0 \Rightarrow 4 = 3\,h \Rightarrow h = \frac{4}{3}.$$

Now that we know the value of *h* that maximizes the volume of the cone, we compute *r*.

$$r = \sqrt{2\,h - h^2} = \sqrt{\frac{8}{3} - \frac{16}{9}} = \sqrt{\frac{24}{9} - \frac{16}{9}} = \sqrt{\frac{8}{9}} = \frac{1}{3}\sqrt{8} = \frac{2}{3}\sqrt{2}.$$

Now let us compute V using these two values.

$$V = \frac{1}{3}\left(\frac{\pi \cdot 8}{9}\right)\cdot\left(\frac{4}{3}\right) = 32\,\frac{\pi}{81}$$

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> evalf((32·Pi/81));
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1.241123024 **(1)**

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So we see that the volume of the largest cone is

$$32\frac{\pi}{81} \approx 1.24.$$

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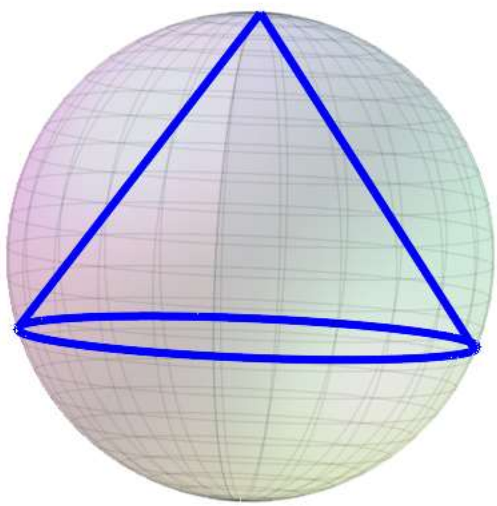

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