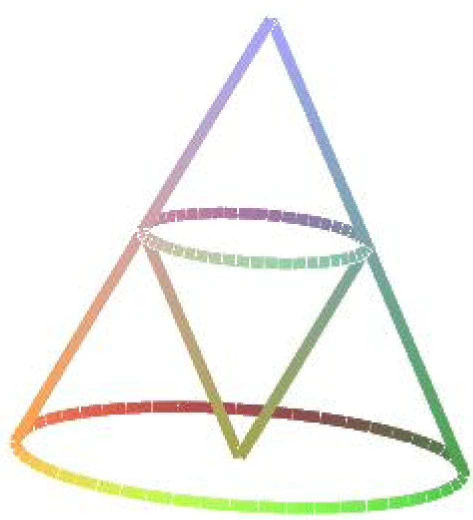
# The Problem of the Month
# March 2021

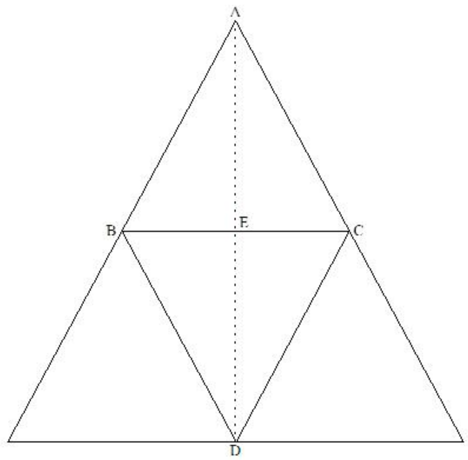
## The Problem:

Imagine that you have two cones, a big one and a smaller one. The big one has a height of $H$ and a radius of its circular base equal to $R$. Your smaller cone has height and radius of circular base equal to $h$ and $r$, respectively. These values satisfy the relation: $H/R = h/r$. Now, take the smaller cone and turn it upside down. Place its vertex at the center of the circular base of the large cone. Doing this yields the following: The circular base of the small inverted cone is tangent to the inside wall of the larger cone. (See figure, below.) Let $V$ stand for the volume of the larger cone and $v$ stand for the volume of the smaller. (Recall that the volume of a cone is equal to one third the area of the base times the height.) Show that $V/v$ is a constant, independent of the values of $H$, $R$, $h$ and $r$. What is this constant?



## The Solution:

Let us consider the cross section of the picture above.




Since *R/H* = *r/h*, it is clear that angles *ABC* and *DBC* must be congruent. Likewise, angles *ACB* and *DCB* must be congruent. Thus, it is obvious that triangle *BCD* is congruent to triangle *ABC* by angle- side - angle.

This means that $2|DE| = 2h = |AD| = H$. Thus $h = H/2$.

Since $H/R = h/r$, we have $r = R/2$.

Now $V = \frac{1}{3}\pi R^2 H$ and $v = \frac{1}{3}\pi r^2 h$

But $v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{R}{2}\right)^2 \frac{H}{2} = \frac{1}{3}\pi\frac{R^2}{4}\frac{H}{2} = \frac{1}{3}\pi R^2 H \times \frac{1}{8} = \frac{1}{8}\times V$

Thus, $\frac{V}{v} = 8$.