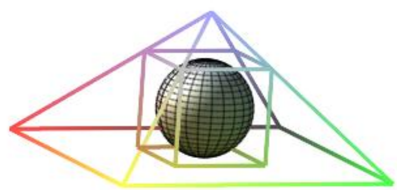
The Problem of the Month
February 2021

The Problem:

Imagine a pyramid with a square $2N$ x $2N$ base and altitude N. Suppose that a cube were placed inside this pyramid so that its upper four corners were situated on the lines where neighboring faces of the pyramid met. Now inscribe a sphere inside this cube. Find the volume of this sphere in terms of N.



The Solution:

The fact that the base is $2N$ x $2N$ and the altitude is N tells us that the angle that each face makes with the base is 45 degrees. Now, let us use this fact to determine the length of the edges of the cube. Take two points, A and B, on opposite edges of the base, directly across from one another. Now move L units from A along the line that connects A and B towards B. Call this new point D. Do the same from the other side. That is move L units from B towards A and call the point arrived at E. Now, the fact that the faces of the pyramid intersect the base at 45 degrees means that the points directly above D and E on the pyramid's faces are L units above D and E, respectively. To make our cube, we want D and E to be L units apart. Thus, we want $2N - 2L = L$. This means that $2N$ must equal $3L$ or

2N/3 = L.

This means that the sphere inscribed in this cube will have radius *N*/3.

Now, recall that the formula for the volume of a sphere is $\frac{4}{3}\pi r^3$. Plugging in *N*/3 for the radius yields a volume of:

$$V = \frac{4}{3}\pi\left(\frac{N}{3}\right)^3 = \frac{4\pi N^3}{81}$$