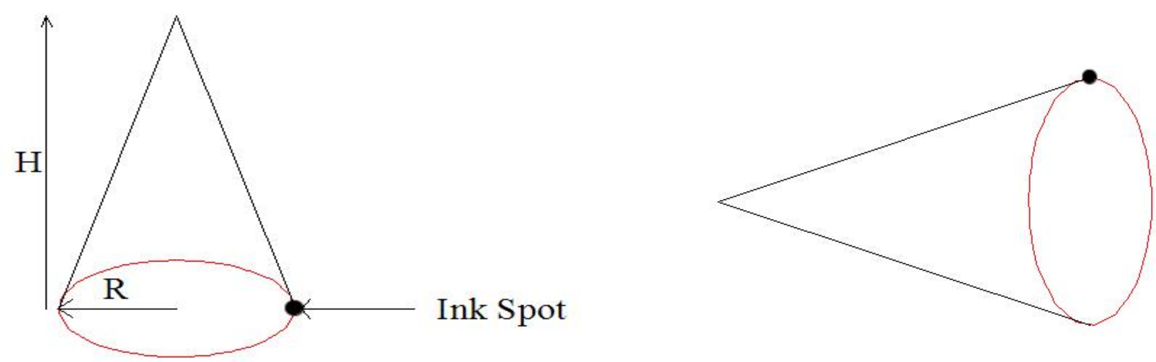
The Problem of the Month
January 2021

The Problem:

Imagine a cone with circular base of radius *R* and height *H*. Suppose that there was a small wet ink spot on the perimeter of the base. Now, place the cone on its side and roll it around on a clean white piece of paper atop a flat surface. The ink spot will produce a dot every time it comes in contact with the paper. Now, given a positive integer, *K*, suppose that you wanted this procedure to produce ***exactly K evenly spaced dots*** around the circle traced out by the base no matter how many times you roll it. Thus, the previously produced ink stains match up with the wet ink, every time, leaving precisely *K* dots. Express *H* as a function of *R* and *K* in order for this to happen.



The Solution:

Clearly, the radius of the circle being traced out by rolling the cone on its side is just the slant height, S, of the cone. S is the hypotenuse of the right triangle with legs R and H. Thus:

$$S = \sqrt{R^2 + H^2}$$

Now, if we want the ink spot to occur at precisely K evenly spaced points around this circle, we must have the circumference of the circle with radius S be precisely K times the circumference of the base of the cone, i.e. a circle of radius R.

Hence, we require:

$$\frac{2\pi\sqrt{R^2+H^2}}{2\pi R} = K$$

Thus:

$$\sqrt{R^2 + H^2} = KR$$

Hence:

$$H = R\sqrt{K^2 - 1}$$