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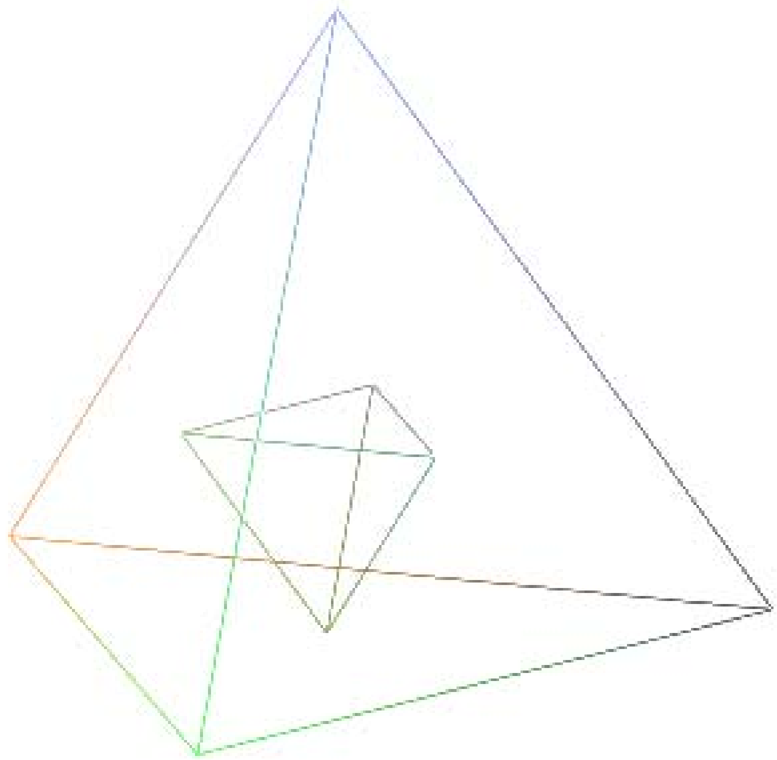
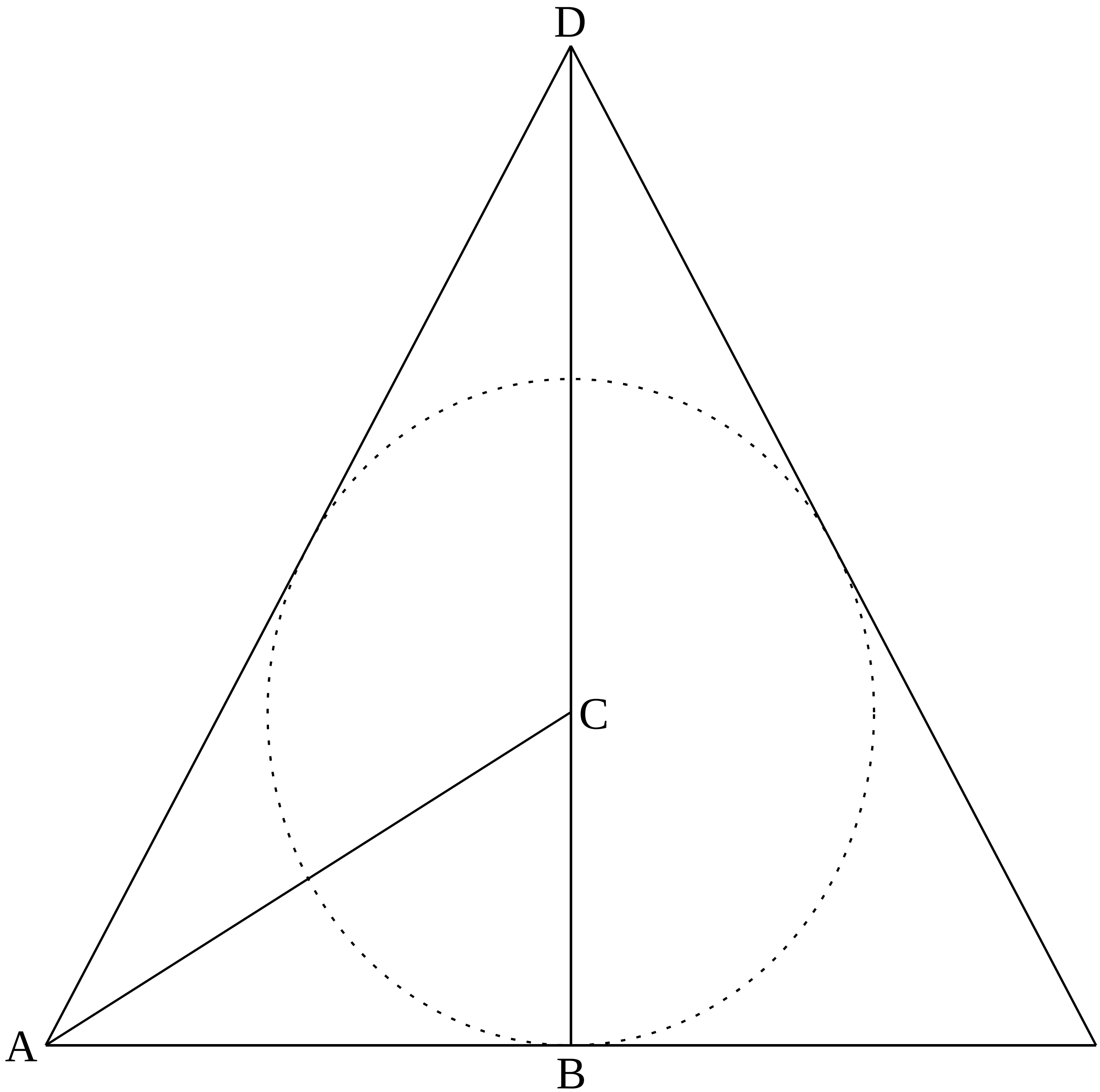
MAA Problem of the Month for December 2020

The Problem: The regular tetrahedron is one of the five Platonic solids. It has four faces, all identical equilateral triangles. It also has four vertices. As such, a small regular tetrahedron can be inscribed inside a larger one with the four vertices of the small tetrahedron placed at the in-centers of the four faces of the larger one. See the picture below. If the large tetrahedron has all of its edges of length S, find the volume contained in the larger tetrahedron but outside the smaller inscribed one. Express this as a function of S and evaluate it when S = 1.



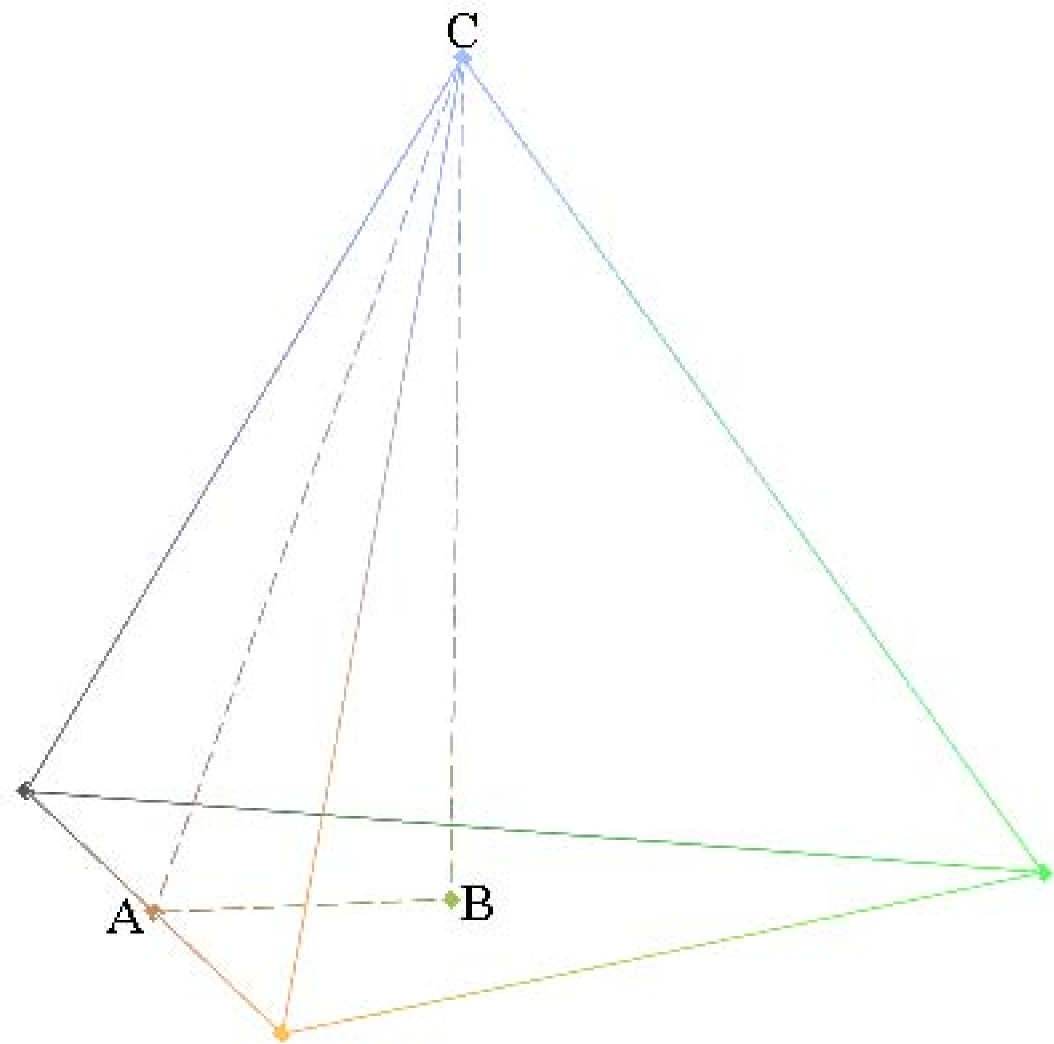
The Solution:   Let us begin with an equilateral triangle with all sides of length S.




Here, B is the mid-point of an edge and C is the in-center. Simple trigonometry gives:

$$|BC| = \frac{S}{2\cdot\sqrt{3}} \qquad |AC| = \frac{S}{\sqrt{3}} \qquad |BD| = \frac{\sqrt{3}S}{2}$$

Now, let us consider a tetrahedron with all edges of length S.

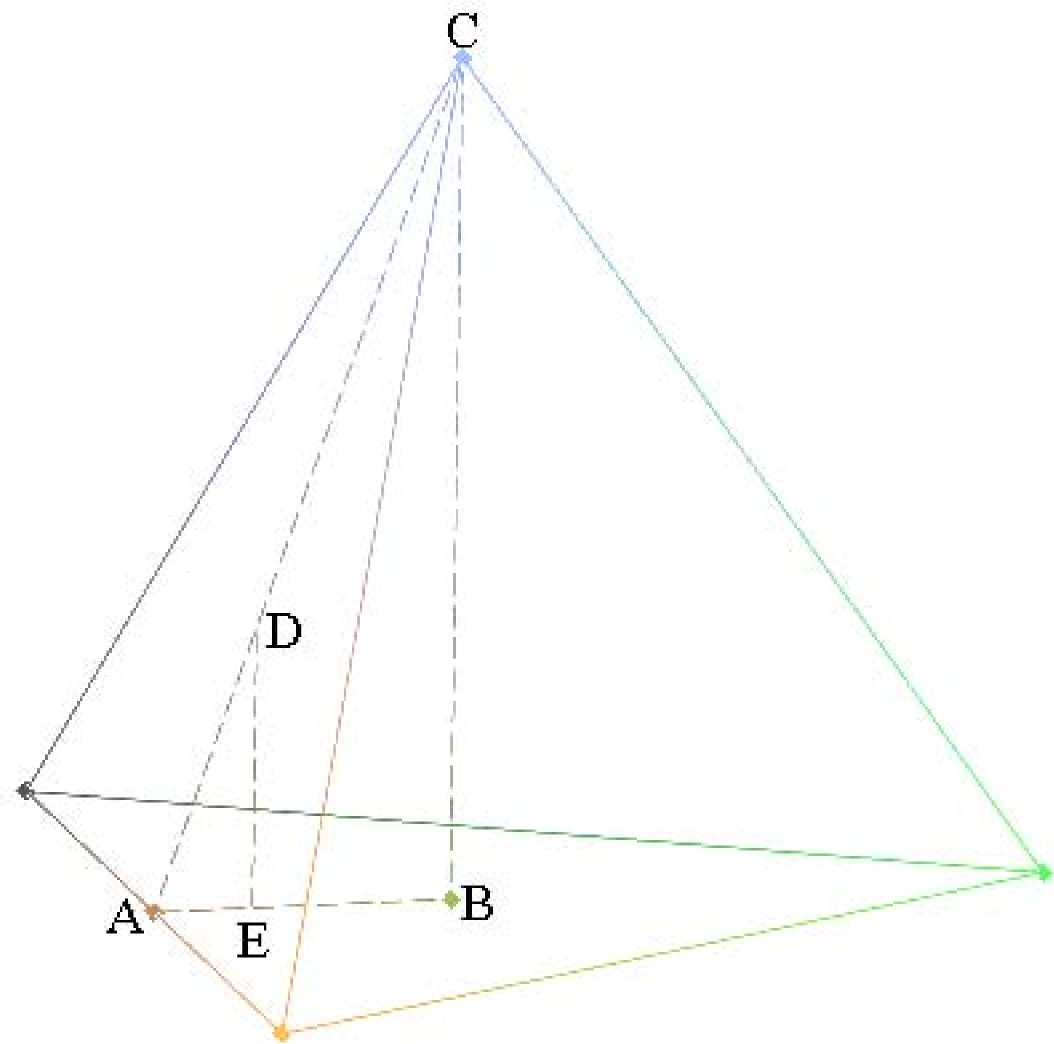

C
A
B

Here we have dropped a perpendicular from the apex, C, to B, the in-center of the horizontal base. (Yes, the in-center of the base is directly below the apex.) From our discussion above, we know that $|AB| = \frac{S}{2\sqrt{3}}$ and $|AC| = \frac{\sqrt{3}\,S}{2}$. Now the Pythagorean Theorem tells us the the altitude of the tetrahedron is

$|BC| = \sqrt{\frac{2}{3}}\,S.$ Let us call this value, H.

Now that we have a relationship between the length of the edges of a regular tetrahedron and its altitude, let us solve it for S in terms of H: $S = \sqrt{\frac{3}{2}}\,H.$

Now, let us go up $\frac{S}{2\sqrt{3}}$ along the line segment AC. This point, D, will be the in-center of the slanted face. Now, drop a perpendicular from this point to the horizontal base at E. See diagram below.


C
D
A
E
B

Observe that triangle ADE is similar to triangle ACB. Thus,

$$\frac{|AD|}{|DE|} = \frac{|AC|}{|BC|}$$

We know the value of three of the four terms:

$$\frac{\left(\frac{S}{2\sqrt{3}}\right)}{|DE|} = \frac{\left(\frac{\sqrt{3}S}{2}\right)}{\sqrt{\frac{2}{3}}\,S}$$

Let us solve this for |DE|. We get |DE| $= \frac{1}{3}\sqrt{\frac{2}{3}}S$. Call this value h, the altitude of the small inscribed tetrahedron. Recalling our relationship between the length of a side of a tetrahedron and its altitude, $S = \sqrt{\frac{3}{2}}\,H$, we plug h into this formula and find that the sides of the small tetrahedron are exactly 1/3 the length of the sides of the large tetrahedron.

Now, let us put this all together. The base of the large tetrahedron is $\frac{\sqrt{3}S^2}{4}$. The altitude of the large tetrahedron is $\sqrt{\frac{2}{3}}S$. The volume of a tetrahedron (like that of any cone) is 1/3 ×area of base × height. Thus the volume of the large tetrahedron is $\frac{S^3}{6\sqrt{2}}$.

The base of the small tetrahedron is $\frac{\sqrt{3}S^2}{36}$ and its altitude is $\frac{1}{3}\sqrt{\frac{2}{3}}S$. Thus, the volume of the small tetrahedron is $\frac{S^3}{162\sqrt{2}}$

. Not surprisingly, the volume of the small tetrahedron is $\frac{1}{27}$ times the volume of the large.

Now, the volume that we seek is

$$\frac{S^3}{6\sqrt{2}} - \frac{S^3}{162\sqrt{2}} = \frac{S^3}{\sqrt{2}}\left(\frac{27}{162} - \frac{1}{162}\right) = \frac{S^3}{\sqrt{2}}\left(\frac{26}{162}\right) =$$

$$\boldsymbol{\frac{S^3}{\sqrt{2}}\left(\frac{13}{81}\right)}$$

When S = 1, this value is approximately 0.1135