MAA Problem of the Month for December 2020

The Problem: The regular tetrahedron is one of the five Platonic solids. It has four faces, all identical equilateral triangles. It also has four vertices. As such, a small regular tetrahedron can be inscribed inside a larger one with the four vertices of the small tetrahedron placed at the in-centers of the four faces of the larger one. See the picture below. If the large tetrahedron has all of its edges of length $S$, find the volume contained in the larger tetrahedron but outside the smaller inscribed one. Express this as a function of $S$ and evaluate it when $S = 1$. 
The Solution: Let us begin with an equilateral triangle with all sides of length $S$. 
Here, B is the mid-point of an edge and C is the in-center. Simple trigonometry gives:

\[ |BC| = \frac{S}{2 \cdot \sqrt{3}} \quad |AC| = \frac{S}{\sqrt{3}} \quad |BD| = \frac{\sqrt{3}S}{2} \]
Now, let us consider a tetrahedron with all edges of length $S$. 
Here we have dropped a perpendicular from the apex, C, to B, the in-center of the horizontal base. (Yes, the in-center of the base is directly below the apex.) From our discussion above, we know that

\[ |AB| = \frac{S}{2\sqrt{3}} \quad \text{and} \quad |AC| = \frac{\sqrt{3}}{2} S. \]

Now the Pythagorean Theorem tells us the altitude of the tetrahedron is

\[ |BC| = \sqrt{\frac{2}{3}} S. \]

Let us call this value, H.

Now that we have a relationship between the length of the edges of a regular tetrahedron and its altitude, let us solve it for S in terms of H:

\[ S = \sqrt{\frac{3}{2}} H. \]

Now, let us go up \( \frac{S}{2\sqrt{3}} \) along the line segment AC. This point, D, will be the in-center of the slanted face. Now, drop a perpendicular from this point to the horizontal base at E. See diagram below.
Observe that triangle ADE is similar to triangle ACB. Thus,
\[
\frac{|AD|}{|DE|} = \frac{|AC|}{|BC|}
\]

We know the value of three of the four terms:
\[
\left( \frac{S}{2\sqrt{3}} \right) = \left( \frac{\sqrt{3}S}{2} \right) \quad \text{Let us solve this for } |DE|. \quad \text{We get } |DE| = \frac{1}{3} \sqrt{\frac{2}{3}} S.
\]

Call this value h, the altitude of the small inscribed tetrahedron. Recalling our relationship between the length of a side of a tetrahedron and its altitude, \(S = \sqrt{\frac{3}{2}} H\), we plug h into this formula and find that the sides of the small tetrahedron are exactly 1/3 the length of the sides of the large tetrahedron.

Now, let us put this all together. The base of the large tetrahedron is \(\frac{\sqrt{3}S^2}{4}\). The altitude of the large tetrahedron is \(\sqrt{\frac{2}{3}} S\). The volume of a tetrahedron (like that of any cone) is \(1/3 \times \text{area of base} \times \text{height}\). Thus the volume of the large tetrahedron is \(\frac{S^3}{6\sqrt{2}}\).

The base of the small tetrahedron is \(\frac{\sqrt{3}S^2}{36}\) and its altitude is \(\frac{1}{3} \sqrt{\frac{2}{3}} S\). Thus, the volume of the small tetrahedron is \(\frac{S^3}{162\sqrt{2}}\).
Not surprisingly, the volume of the small tetrahedron is \( \frac{1}{27} \) times the volume of the large.

Now, the volume that we seek is

\[
\frac{S^3}{6\sqrt{2}} - \frac{S^3}{162\sqrt{2}} = \frac{S^3}{\sqrt{2}} \left( \frac{27}{162} - \frac{1}{162} \right) = \frac{S^3}{\sqrt{2}} \left( \frac{26}{162} \right) =
\]

\[
\frac{S^3}{\sqrt{2}} \left( \frac{13}{81} \right)
\]

When \( S = 1 \), this value is approximately \( 0.1135 \).