Problem of the Month for November 2020

Start with a unit cube and inscribe a sphere. Then inscribe 8 small spheres in the empty corners so that they are tangent to the sides of the cube and, also, to the large central sphere. Find the volume of the region inside the cube and outside all of the spheres.

First of all, the central sphere will have equation $x^2 + y^2 + z^2 = \frac{1}{4}$. The points on this sphere that will be tangent to the 8 small spheres will have coordinates

$$\left(\pm \frac{1}{2\sqrt{3}}, \pm \frac{1}{2\sqrt{3}}, \pm \frac{1}{2\sqrt{3}} \right).$$

To find the radius of each of the small spheres, we set the distance from the side of the cube equal to the distance to the point of tangency with the central sphere. Here we have taken the cube and the central sphere to have center at the origin so the sides of the cube will be the planes given by $x = \pm\frac{1}{2}$, $y = \pm\frac{1}{2}$ & $z = \pm\frac{1}{2}$.

Thus: $$\frac{1}{2} - x = \sqrt{3\left(x - \frac{1}{2\sqrt{3}}\right)^2} = \sqrt{3}\left(x - \frac{1}{2\sqrt{3}}\right) = \sqrt{3}x - \frac{1}{2}$$

$$1 = \sqrt{3}x + x = x\left(1 + \sqrt{3}\right) \Rightarrow x = \frac{1}{1+\sqrt{3}}.$$

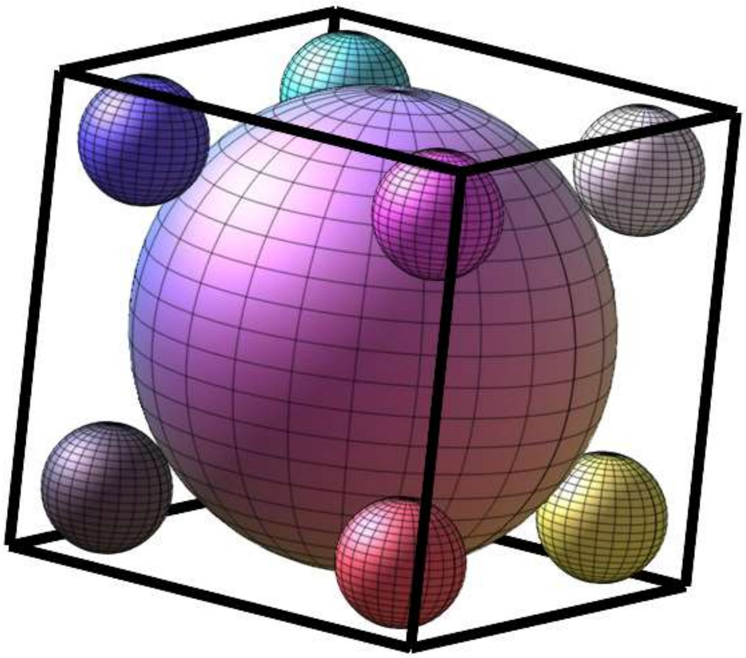
Thus, the small sphere that lies on the line $x = t \quad y = t \quad z = t$ will have

center $= \left(\frac{1}{1+\sqrt{3}}, \frac{1}{1+\sqrt{3}}, \frac{1}{1+\sqrt{3}} \right)$ and radius $\frac{1}{2} - \frac{1}{1+\sqrt{3}}$.

The other 7 small spheres will all have the same radius and centers with the same coordiantes in absolute values.

>

> *display*([*a*, *b*, *c*, *dd*, *e*, *f*, *g*, *h*, *i*, *j*, *k*, *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, *t*, *u*], *axes* = *none*);



>

Now let us compute the volume contained inside the cube and outside the 9 spheres. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. We have 8 spheres with radius $\frac{1}{2} - \frac{1}{1+\sqrt{3}}$ and one large sphere with radius $\frac{1}{2}$. So the total volume of the spheres is:

$$\frac{4}{3}\pi\cdot\left(\frac{1}{2} - \frac{1}{1+\sqrt{3}}\right)^3 \times 8 + \frac{4}{3}\pi\cdot\left(\frac{1}{2}\right)^3$$

So the volume we seek is:

$$1 - \left(\frac{4}{3}\pi\cdot\left(\frac{1}{2} - \frac{1}{1+\sqrt{3}}\right)^3 \times 8 + \frac{4}{3}\pi\cdot\left(\frac{1}{2}\right)^3 \right)$$

> $1 - \left(\frac{4}{3}\cdot\text{Pi}\cdot\left(\frac{1}{2} - \frac{1}{1+\text{sqrt}(3)}\right)^3\cdot 8 + \frac{4}{3}\cdot\frac{\text{Pi}\cdot 1}{8}\right);$

$$1 - \frac{32\,\pi\left(\frac{1}{2} - \frac{1}{1+\sqrt{3}}\right)^3}{3} - \frac{\pi}{6} \quad \textbf{(1)}$$

> *simplify*(%);

$$\frac{(-27\,\pi + 18)\sqrt{3} + 35\,\pi + 30}{3\left(1+\sqrt{3}\right)^3} \quad \textbf{(2)}$$

> *evalf*(%);

$$0.3958177536 \quad \textbf{(3)}$$