

Talk on calculus teaching – Taylor polynomials and numerical integration, Spring 2026

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Introduction

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- I also believe students are empowered when they figure important results out or even if they try to figure them out and only partially succeed.
- I hope you do try this out (professors) or think this through (students).

BACKGROUND INFORMATION

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As well, the difference quotient has the limit $f'(a)$ as $x \rightarrow a$. Justified by either: definition of derivative (professors) or L'Hopital's rule (students).

GRAPHICS LARGER SCALE

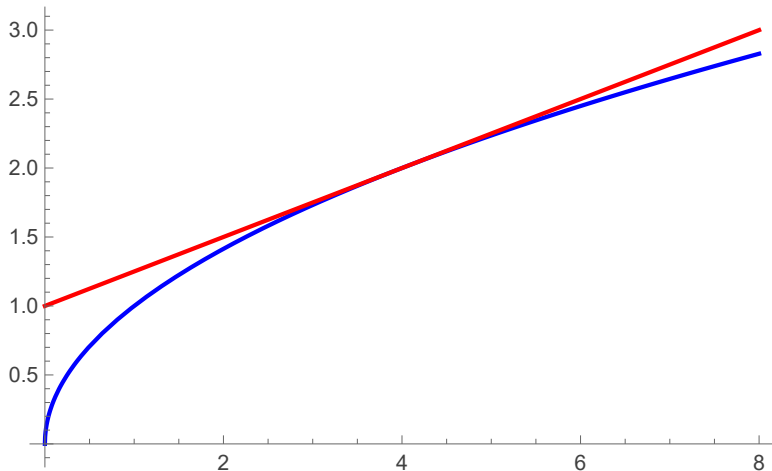


Figure: square root and linear approximation

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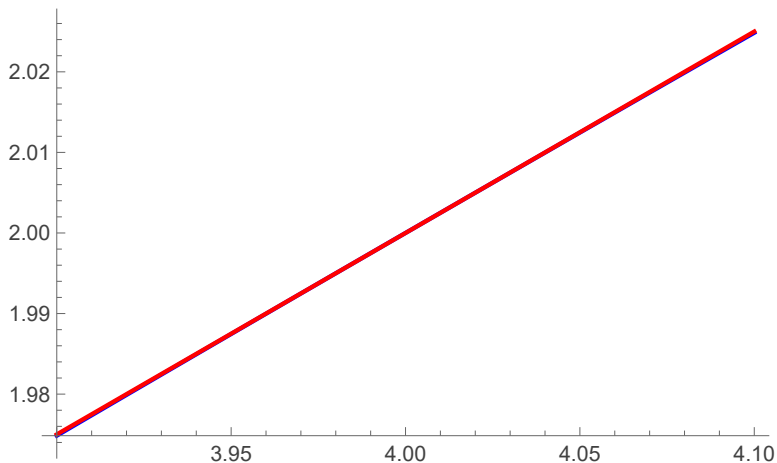


Figure: square root and linear approximation zoomed in

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Where have you seen this in your course? Did you use it to discuss convexity? Second derivative test? How does this generalize?

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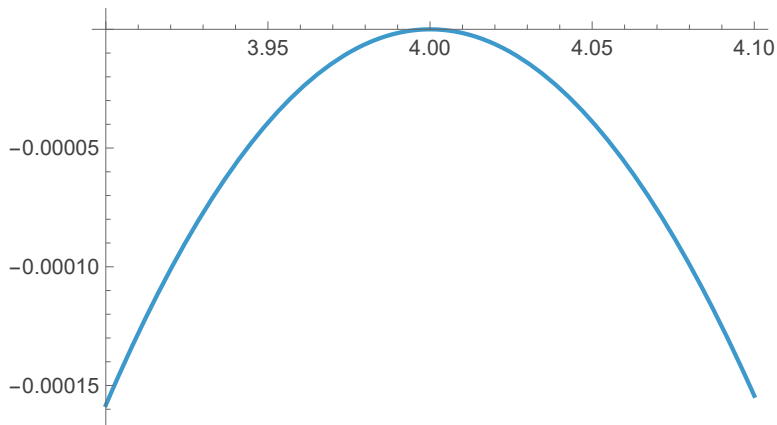


Figure: square root minus linear approximation zoomed in

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On to numerical integration ideas!

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Key fact used: integration adds up pieces, can add many small pieces!

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Exact result for quadratic uses blend of midpoint (0) and trapezoid (h^2) values, in a 2 : 1 ratio.

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The summed remainder or residual or error on the full original interval uses values of some derivative of f summed on each piece. Books usually reference the maximal value of f'' or $f^{(4)}$ but the error really has something like the average value of such terms.

$$F(h) = F(0) + F'(0)h + \frac{1}{2}F''(0)h^2 + \frac{1}{6}F^{(3)}(0)h^3 + \dots$$

$$F(-h) = F(0) - F'(0)h + \frac{1}{2}F''(0)h^2 - \frac{1}{6}F^{(3)}(0)h^3 + \dots$$

so upon subtraction we see:

$$\int_{-h}^h f(x) dx = 2hF'(0) + \frac{2}{6}F^{(3)}(0)h^3 + \dots$$

and we note that the first term is the midpoint rule so the remaining terms are the residual.

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$$2h\left(\frac{f(-h) + f(h)}{2}\right) = 2h(f(0) + \frac{1}{2}f''(0)h^2 + \dots)$$

to compare with

$$\int_{-h}^h f(x) dx = 2hF'(0) + \frac{2}{6}F^{(3)}(0)h^3 + \dots$$

which again creates a residual expression after comparing, again involving h^3 as leading term.

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To go from the sums with higher terms involved to exact statements with equal signs in any of these analyses again requires mean value theorem ideas, which means evaluating some derivative at a location on our given interval, typically not at the origin (our midpoint). And then summing to get the full integral of interest.

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Thanks for your attention! Happy to answer questions.