Affine Type C_n Curve Neighborhoods

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Outline

- Background
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- Affine definitions
- Affine type $\widetilde{A}_n$
- Affine type $\widetilde{C}_n$

Some Setup

G	Complex semisimple Lie algebra
$B \subset G$	Borel subgroup
$X = G/B$	Homogeneous space
$\Omega \subset X$	Schubert variety
$d \in H_2(X)$	Effective degree of a curve
$\Gamma_d(\Omega)$	(Closure of the) union of rational curves of degree d which intersect Ω
W	Weyl group
Π	Root system

Geometry $\to$ Combinatorics

With this setup, there is a bijection,

$$\text{Schubert varieties } \Omega \quad \leftrightarrow \quad \text{Weyl group elements } w \in W$$

We label the variety associated with w by $X(w)$.

Hecke Products

The Weyl group W comes with its length function.
In type A_{n-1}: $W = S_n$, with the usual length of a symmetric group element.
Def: The Hecke product on W, with a simple reflection s_α, is given by:

$$w \cdot s_\alpha = \begin{cases} ws_\alpha & \text{if} \quad \ell(ws_\alpha) > \ell(w) \\ w & \text{otherwise} \end{cases}$$

This product only goes 'up' in the Bruhat order.

Non-Affine (finite) Types

Buch-Mihalcea construction: curve neighborhoods can be found using a recurrence- if $\alpha^\vee \leq d$, then

$$\Gamma_d(X(w)) = \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha))$$

Everything can be built from the curve neighborhood at the identity,

$$\Gamma_d(X(id))$$

which we call $X(z_d)$ for some special $z_d \in W$.

Affine Type A

Affine roots: $\Pi = \{\alpha + k\delta \mid \alpha \in W_f, k \in \mathbb{Z}\}$

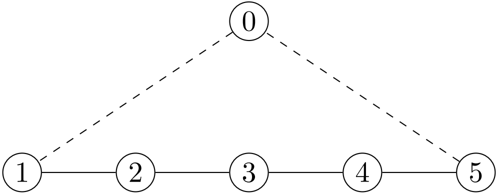
Affine roots have an 'imaginary' part, δ.

The good news: The recurrence relation still works! We can still reduce to finding the curve neighborhood at the identity.

The bad news: δ is not itself a root, but the degree $d = \delta = \alpha_0 + \alpha_1 + \ldots + \alpha_n$ is an effective degree. This leads to multiple Weyl group elements reachable by this degree.

Other good news: We can 'build up' to any roots from the imaginary root (and its multiples).

Making affine type A simpler: diagram isomorphisms



$$\Gamma_\delta = \{t_\gamma \mid \gamma \in \Pi\}$$

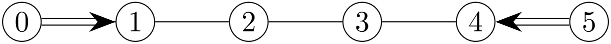
Or, more simply, $\Gamma_\delta = \{s_{\alpha^\vee} \cdot s_{\delta-\alpha^\vee} \mid \alpha \in \Pi^+\}$

Affine Type C

Affine Type C features:

- There are short and long roots - root vs. coroot distinction matters.
- The affine roots still look like $\alpha + k\delta$ for $\alpha \in W_f$
- We still have an imaginary root problem, $\Gamma_\delta(X(id))$.

Type C Dynkin Diagram



We can still use a symmetry of the Dynkin diagram to reduce to the finite case if $d_i = 0$ for any degree component.

Conjecture (In progress)

- $\Gamma_\delta = \{t_\gamma \mid \gamma \in \Pi\}$
- $|\Gamma_\delta| = 2n^2$

References

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