Finitely Additive Measures in Number Theory

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We use these words to make it appear that we're studying other things. They are all some analog of prime.

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2. Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$, i.e., $\overline{\mathbb{Q}}$ is a smallest field containing the roots of all polynomials with rational coefficients.
   - What do primes look like in $\overline{\mathbb{Q}}$?
   - Does it even make sense to speak of primes here?
   - If so, how many primes should I expect to see?

Absolute Values on Fields

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Let F be a field. An *absolute value* on F is a function $|\cdot| : F \to [0, \infty)$ which satisfies the following properties:

1. $|x| = 0$ if and only if $x = 0$
2. $|xy| = |x| \cdot |y|$ for all $x, y \in F$
3. $|x + y| \leq |x| + |y|$ for all $x, y \in F$ (Triangle Inequality).

The Trivial Absolute Value

Every field field has at least one absolute value given by

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- If F is a finite field, then the trivial absolute value is the only absolute value on F.
- Otherwise, we can expect to see many other types.

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When creating new absolute values on a field, we want to avoid this silly technique.

- Two absolute values $\|\cdot\|$ and $|\cdot|$ are called *equivalent* if there exists $\theta > 0$ such that $\|x\| = |x|^{\theta}$ for all $x \in F$.
- An equivalence class of non trivial absolute values on F is called a *place* of F.

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It is possible to prove that none of the above absolute values are equivalent.

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Theorem 1 (Ostrowski).

Every absolute value on $\mathbb{Q}$ is equivalent to one of the following:

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We no longer think of a prime as an element of $\mathbb{Z}$, but rather, we interpret it as a place of $\mathbb{Q}$.

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- If K is a number field and v is a place of K, write $Y(K, v)$ to denote the set of places of $\overline{\mathbb{Q}}$ that divide v.
- The collection of all sets of the form $Y(K, v)$ forms a basis for a topology on Y.

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of places of $\overline{\mathbb{Q}}$ is a disjoint countable union of Cantor sets. As with number fields, the above discussion should be seen as a description of the primes of $\overline{\mathbb{Q}}$.

The Prime Counting Homomorphism

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We can gain some insight by doing a little measure theory on Y.

Rings of Sets

We shall fix a set S of places of $\mathbb{Q}$ and let

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It follows from these assumptions that $\emptyset \in \mathcal{R}$, and from De Morgan's laws, that $A \cap B \in \mathcal{R}$.

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- $\mathcal{R}$ is precisely the collection of open compact subsets of X.
- $\mathcal{R}$ is a ring of sets on X.

Finitely Additive Measures

A map $\mu : \mathcal{R} \to \mathbb{R}$ is called a *measure* on $\mathcal{R}$ if

(i) $\mu(\emptyset) = 0$

(ii) If $A, B \in \mathcal{R}$ are disjoint sets then $\mu(A \cup B) = \mu(A) + \mu(B)$.

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If you want more precision, you might refer to my definition as a *finite-valued finitely-additive signed measure on* $\mathcal{R}$.

Extensions of Ω

Equipped with this definition of measure, we can create many extensions of Ω to $\overline{\mathbb{Q}}$ using the following easy steps:

1. Let $S = \{2, 3, 5, 7, 11, \ldots\}$ and let X be the set of places of $\overline{\mathbb{Q}}$ that divide a place in S.

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3. Let K be a number field and let S_K be the set of places of K dividing a place in S.

Given a non-zero point $\alpha \in K$, we define

$$\Omega(\alpha) = \sum_{v \in S_K} \frac{\log |\alpha|_v}{\log p_v} \cdot \mu(Y(K, v)).$$

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- If $\alpha \in \mathbb{Q}^\times$ then we may write $\alpha = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and we find

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Hence, our new definition of $\Omega(\alpha)$ agrees with our previous definition when $\alpha \in \mathbb{Q}$.

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- There is a unique measure λ that causes Ω to give equal values to all pairs of Galois conjugates over $\mathbb{Q}$. Using this measure

$$\Omega(1+i) = \Omega(1-i) = \frac{1}{2}.$$

An Vector Space Related to $\overline{\mathbb{Q}}$

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is a vector space over $\mathbb{Q}$ with addition and scalar multiplication given by

$$(\alpha, \beta) \mapsto \alpha\beta \quad \text{and} \quad (r, \alpha) = \alpha^r.$$

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is a vector space over $\mathbb{Q}$ with addition and scalar multiplication given by

$$(\alpha, \beta) \mapsto \alpha\beta \quad \text{and} \quad (r, \alpha) = \alpha^r.$$

$\Omega : \mathcal{V} \to \mathbb{Q}$ is a well-defined linear transformation, i.e., it is an element of the algebraic dual of $\mathcal{V}$.

An Vector Space Related to $\overline{\mathbb{Q}}$

This connection between measures and Ω is a special case of a more general result.

Let $\overline{\mathbb{Z}}$ be the ring of algebraic integers. Then quotient space

$$\mathcal{V} := \overline{\mathbb{Q}}^{\times}/\overline{\mathbb{Z}}^{\times}$$

is a vector space over $\mathbb{Q}$ with addition and scalar multiplication given by

$$(\alpha, \beta) \mapsto \alpha\beta \quad \text{and} \quad (r, \alpha) = \alpha^r.$$

$\Omega : \mathcal{V} \to \mathbb{Q}$ is a well-defined linear transformation, i.e., it is an element of the algebraic dual of $\mathcal{V}$.

Theorem 2 (S, 2022).

The space of rational valued measures is isomorphic to the algebraic dual of $\mathcal{V}$.

The End