Generalizations for Maxwell's Equations to Yang-Mills Equations

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Maxwell's Equations for $\mathbb{R}^4$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t}$$

$$= \vec{J}$$

Differential Forms

Recall that the directional derivative in $\mathbb{R}^n$ in the direction v is

$$\nabla f \ \cdot v = vf$$

The gradient 'keeps track' of the directional derivatives of f in all directions. We want something that does the same thing on any manifold. This leads us to the idea of differential forms!

1-forms

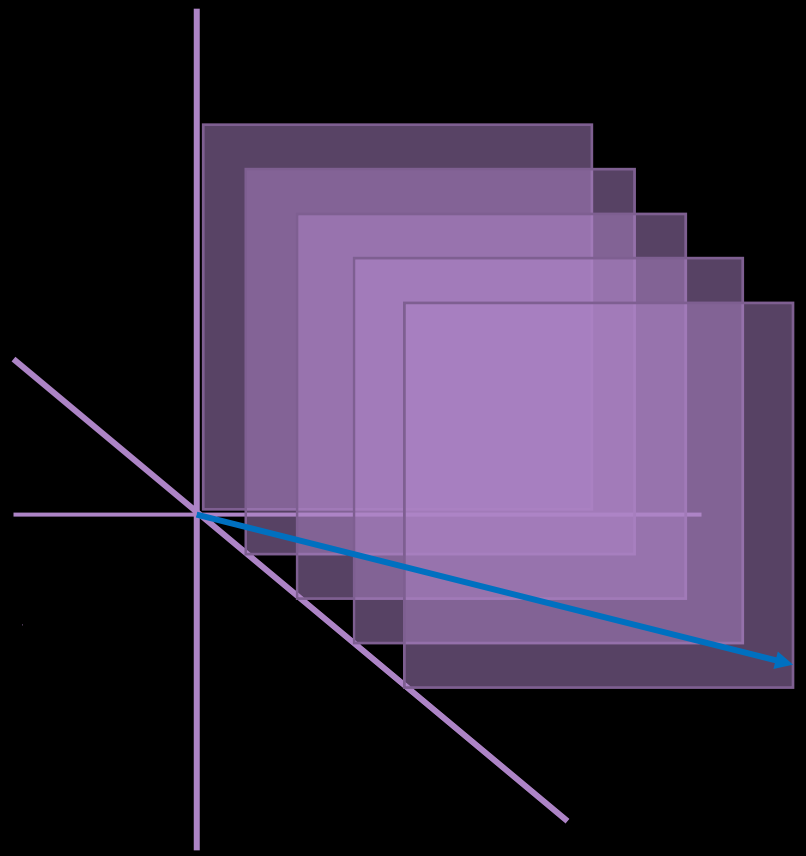
We define a 1-form to be a map on any manifold M

$$\omega\colon \mathrm{Vect(M)} \mapsto C^{\infty}(M)$$

that is linear over $C^{\infty}(M)$.

We can think of a 1-form “eating” a vector and the spitting out the number of planes the vector pierces

The space of 1-forms is called $\Omega^1(M)$



Exterior Derivative

We can now define the 1-form df, where $v \in \text{Vect(M)}$ and $f \in C^{\infty}(M)$ as:

$$df(v) = vf \ (1)$$

We call df the differential of f or the exterior derivative of f. This is the analog of the gradient!

The below map is also called the differential, as it sends each function f to its differential df.

$$d: C^{\infty}(M) \to \Omega^1(M)(2)$$

Exterior Derivative

Let $f, g \in C^{\infty}(M)$ and $v, w \in \text{Vect(M)}$. We can show that (1) really is a 1-form by checking linearity:

$$df(v+w) = (v+w)f = vf + wf = df(v) + df(w)$$

$$df(gv) = gvf = gdf(v)$$

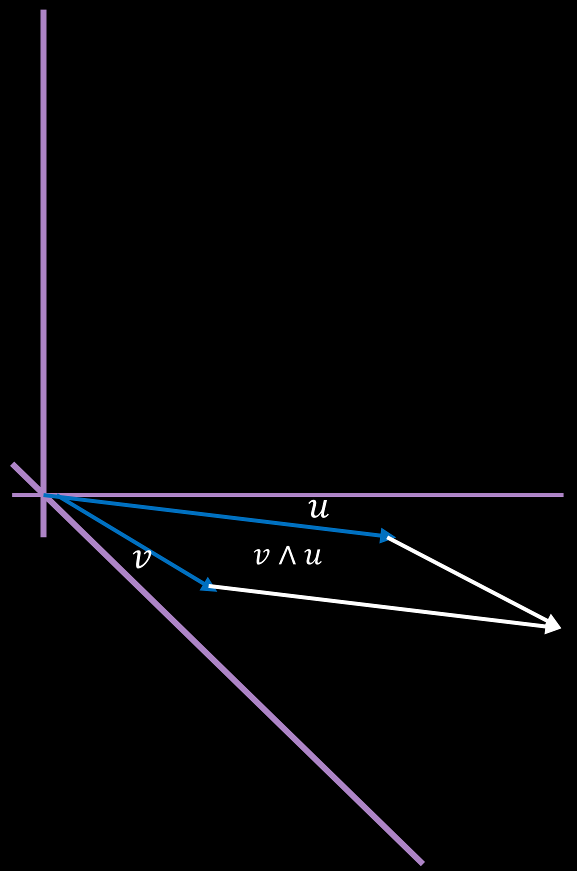
and that (2) satisfies the Leibniz law:

$$d(fg)v = v(fg) = v(f)g + v(g)f = df(v)g + dg(v)f = [fdg + gdf](v)$$

Wedge product

Let V be a vector space. We can generalize the multiplication of vectors with the **exterior algebra** over V, denoted by ΛV, which is the algebra over V with the operation

$$v \wedge w = -w \wedge v \text{ for all } v, w, \in V$$

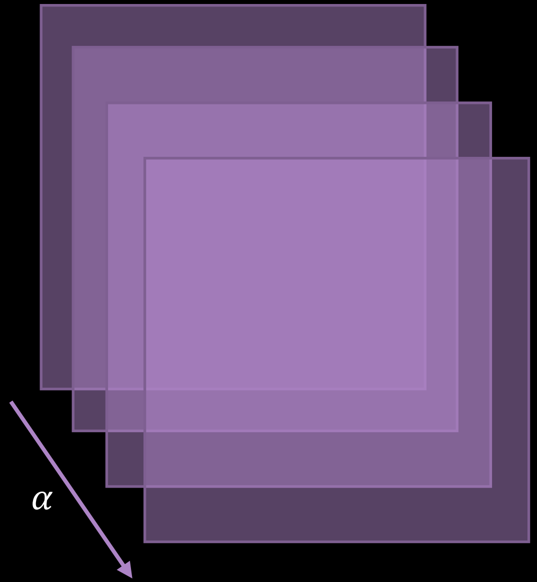


p-forms

By extension, we can form an algebra by taking all the linear combinations of formal products of the form

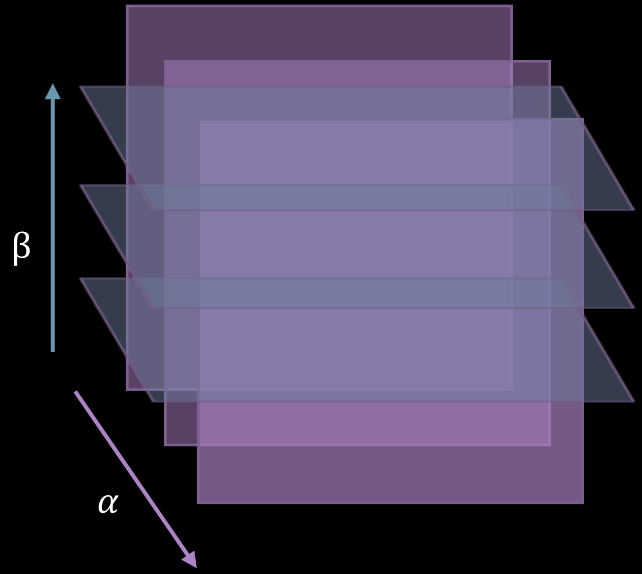
$$v_1 \wedge \cdots \wedge v_n, v_i \in V$$

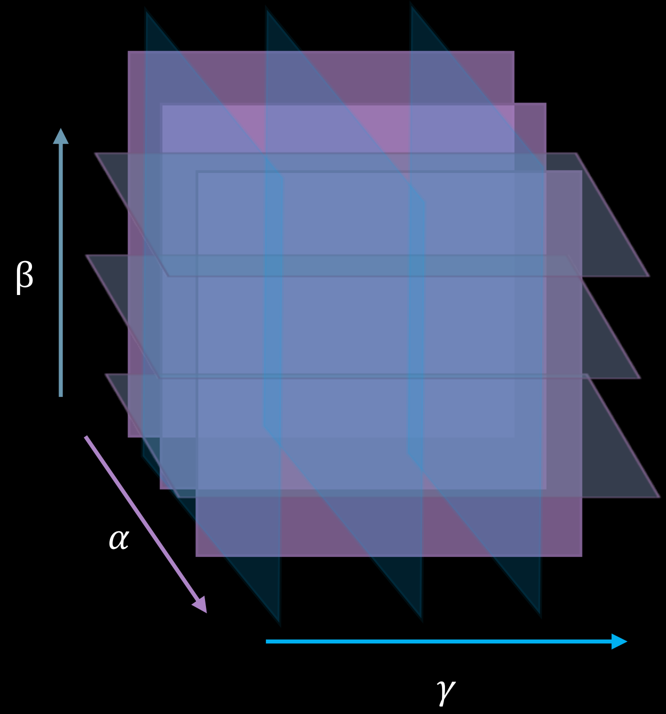
Just like with 1-forms, p-forms are also a map eating a wedge product $v_1 \wedge \cdots \wedge v_p$ and spitting out some combination of projections of areas.



1-form

2-form
$\alpha \wedge \beta$





3-form
$\alpha \wedge \beta \wedge \gamma$

Exterior Derivative of a p-form

We define the exterior derivative, or differential, of a p-form as the unique set of maps

$$d: \Omega^p(M) \to \Omega^{p+1}(M)$$

Such that the following properties hold:

1) $d: \Omega^0(M) \to \Omega^1(M)$ follows our previous definition
2) $d(\omega + \mu) = d\omega + d\mu$ and $d(c\omega) = cd\omega\ \forall \omega, \mu \in \Omega(M)$ and $c \in \mathbb{R}$
3) $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu\ \forall\ \omega \in \Omega^p(M)$ and $\mu \in \Omega(M)$
4) $d(d\omega) = 0\ \forall\ \omega \in \Omega(M)$

Then we can calculate d of any differential form. For example, if we have

$$f dg \wedge dh$$

We can use the rules on the previous slide:

$$\begin{aligned} d(f \mathrm{dg} \wedge dh) &= df \wedge (dg \wedge dh) + f \wedge d(dg \wedge dh) \\ &= df \wedge dg \wedge dh + f d(dg) \wedge dh - f dg \wedge d(dh) \\ &= df \wedge dg \wedge dh \end{aligned}$$

This fits the map $d: \Omega^2(M) \rightarrow \Omega^3(M)$

Succinctly, we can write the generalized operations as:

- Gradient: $d: \Omega^0(M) \to \Omega^1(M)$
- Curl: $d: \Omega^1(M) \to \Omega^2(M)$
- Divergence: $d: \Omega^2(M) \to \Omega^3(M)$

The special cases that are used more commonly are when $M = \mathbb{R}^3$

Back to Maxwell's Eq: the first two equations

Recall that the first two equations on $\mathbb{R}^4$ have the form:

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Since we have generalized forms of the divergence and curl, we can use these to rewrite the equations.

The first two equations

The divergence becomes the exterior derivative on 2-forms on $\mathbb{R}^4$ so we treat the magnetic field as a 2-form:

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

The curl becomes the exterior derivative on 1-forms on $\mathbb{R}^4$, so we treat the electric field as a 1-form:

$$E = E_x dx + E_y dy + E_z dz$$

The first two equations

Therefore, the first pair turns into:

$$d_s B = 0$$
$$\partial_t B + d_s E = 0$$

where S is some manifold we call ‘space’ and t is time

The first two equations

Then we can write the unified electromagnetic field F as a 2 form on $\mathbb{R}^4$:

$$F = B + E \wedge dt$$

We can also look at all components:

$$F = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \text{ so,} \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

The first two equations

We can take the exterior derivative of the 2-form:

$$
\begin{aligned}
dF &= dB + dE \wedge dt \\
&= d_s B + dt \wedge \partial_t B + (d_s E + dt \wedge \partial_t E) \wedge dt \\
&= d_s B + (\partial_t B + d_s E) \wedge dt
\end{aligned}
$$

Which results in $dF = 0$ since

$$
d_s B = 0
$$

$$
\partial_t B + d_s E = 0
$$

Hence, we achieve a simple equation that encapsulates the first two equations:

$$dF = 0$$

Hodge Star Operator

For this operator, we need a metric and an orientation. Let M be an n-dimensional oriented semi-Riemannian manifold. Then the inner product of two p forms ω and μ on M is a function $\langle \omega, \mu \rangle$ on M.

In general, we define the Hodge star operator

$$\star: \Omega^p(M) \to \Omega^{n-p}(M)$$

To be the unique linear map from p-forms to $n-p$-forms such that $\forall \omega, \mu \in \Omega^p(M)$,

$$\omega \wedge \star\, \mu = \langle \omega, \mu \rangle \mathrm{vol}$$

where vol is the volume form, defined by $\sqrt{|det g_{\mu\nu}|}dx^1 \dots dx^n$

Hodge Star Operator on basis elements of $\mathbb{R}^3$

On the 0-form:

$$\star\, 1 = dx \wedge dy \wedge dz$$

On 1-forms:

$$\star\, dx = dy \wedge dz \qquad \star\, dy = dz \wedge dx \qquad \star\, dz = dx \wedge dy$$

On 2-forms:

$$\star\, dy \wedge dz = dx \qquad \star\, dz \wedge dx = dy \qquad \star\, dx \wedge dy = dz$$

On the 3-form:

$$\star\, dx \wedge dy \wedge dz = 1$$

Hodge Star on differential forms in $\mathbb{R}^3$

Let ω and μ be 1-forms

$$\omega = \omega_i dx^i \qquad\qquad \mu = \mu_i dx^i$$

Recall that we need a metric and an orientation to define the Hodge Star operator. Using the standard metric, we obtain the 1-form

$$\star\,(\omega \wedge \mu) = (\omega_y \mu_z - \omega_z \mu_y)dx + (\omega_z \mu_x - \omega_x \mu_z)dy + (\omega_x \mu_y - \omega_y \mu_x)dz$$

This is exactly the cross product in $\mathbb{R}^3$!

Back to Maxwell's Eq: the second two equations

Recall the Maxwell's Equations:

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{E} = \rho \quad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$

We notice two main differences between the top two and the bottom two: ρ and $\vec{J}$, and that $\vec{B}$ maps to $\vec{E}$ and $\vec{E}$ maps to $-\vec{B}$.

Keep this in mind for the next part.

Recall…

Before looking at the general case, we will first consider M as Minkowski spacetime with a positive orientation and introduce the metric:

$$\eta(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

so that we may define the Hodge star operator.

Also, recall that

$$F = B + E \wedge dt$$

Is the electromagnetic field

The RHS…

We can turn $\vec{j}$ into a 1-form:

$$j = j_1 dx^1 + j_2 dx^2 + j_3 dx^3$$

And then combine ρ and $\vec{j}$ into a single vector field on M

$$\vec{J} = \rho\partial_0 + j^1\partial_1 + j^2\partial_2 + j^3\partial_3$$

Which we can then turn into a 1-form called the current:

$$J = j - \rho dt$$

The LHS...

Taking the dual of F amounts to:

$$E_i \mapsto -B_i \qquad B_i \mapsto E_i$$

$$F = B + E \wedge dt$$

$$\star F = \star B + \star (E \wedge dt)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$\longrightarrow$

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

However, now we have a 2-form on the LHS and a 1-form on the RHS. We can apply the exterior derivative

$$d: \Omega^p(M) \to \Omega^{\mathrm{p+1}}(\mathrm{M})$$

to the 2-form $\star F$, so $d \star F \in \Omega^3(M)$. Now, remembering that

$$\star: \Omega^p(M) \to \Omega^{n-p}(M)$$

Then $\star\, d \star F \in \Omega^1(M)$, which is exactly where we want to be!

For the more general case

Assume that M is any semi-Riemannian manifold that can be written as $M = \mathbb{R} \times S$, where S is space. Also recall that $F = B + E \wedge dt$. From earlier, the first pair of equations is

$$d_s B = 0 \qquad\qquad \partial_t B + d_s E = 0$$

Also suppose that the metric on S is ${}^3g$ and the metric on M is

$$g = -dt^2 + {}^3g$$

Let $\star_S$ be the Hodge star operator on time dependent differential forms on S

Note that the second pair of equations

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$

Can be written as

$$\star_S \, d_S \star_S E = \rho \quad \text{and} \quad -\partial_t E + \star_S \, d_s \star_S B = j$$

Then

$$\star F = \star_S E - \star_S B \wedge dt$$

So,

$$d \star F = \star_S \partial_t E \wedge dt + d_S \star_S E - d_S \star_S B \wedge dt$$

And

$$\star d \star F = -\partial_t E - \star_S d_S \star_S E \wedge dt + \star_S d_S \star_S B$$

Setting $\star d \star F = J$ gives

$$\star_S d_S \star_S E = \rho \quad and \qquad - \partial_t E + \star_S d_S \star_S B = j$$

Which is exactly what we wanted!

To summarize, we have

$$dF = 0 \qquad\qquad \star d \star F = J$$

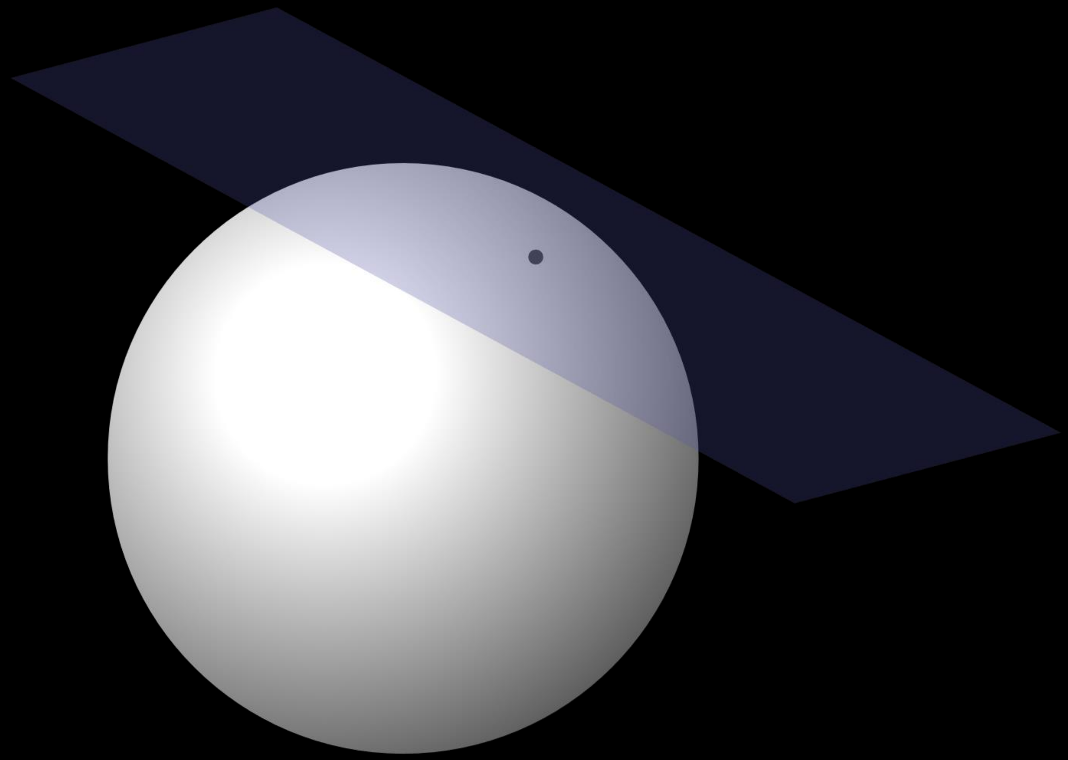
for spacetime manifolds, where F is the electromagnetic field. However, as gauge theory deals with more general fields on spacetime, we have to define these fields and extend these equations.

Bundles

A vector field v on M assigns to each point $p \in M$ a vector in the tangent plane of that point $T_p M$.

So instead of one fixed vector space, we have many vector spaces, which we call a 'bundle'

In order to write differential equations, we need to be able to compare vectors in different vector spaces.

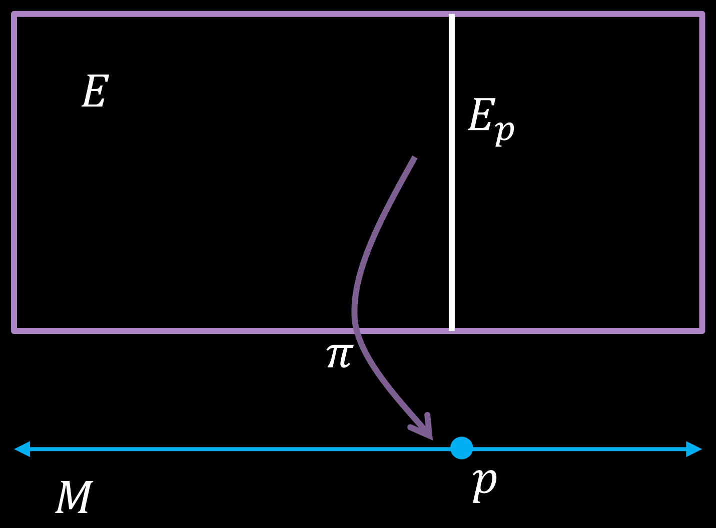


Bundles

A bundle is a structure containing a manifold E, a manifold M, and an onto map $\pi: E \to M$.

We call E the total space, M the base space, and π the projection map.

For each point in p $\in M$, the space $E_p = \{q \in E: \pi(q) = p\}$ is called the fiber over p.

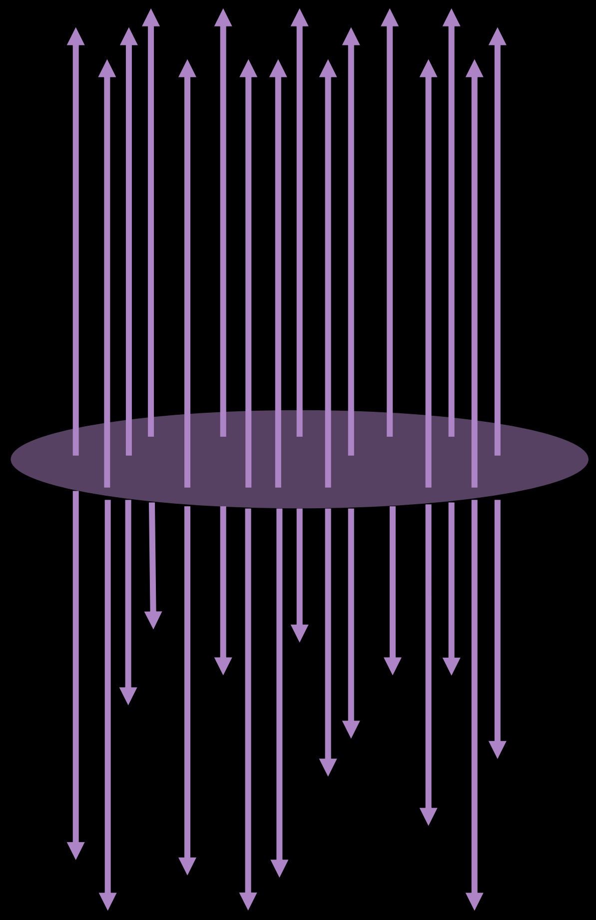


Vector Bundles

We can have two manifolds M and $\mathbb{R}^n$ such that E can be expressed as

$$E = M \times \mathbb{R}^n$$

E here is what is known as a vector bundle!

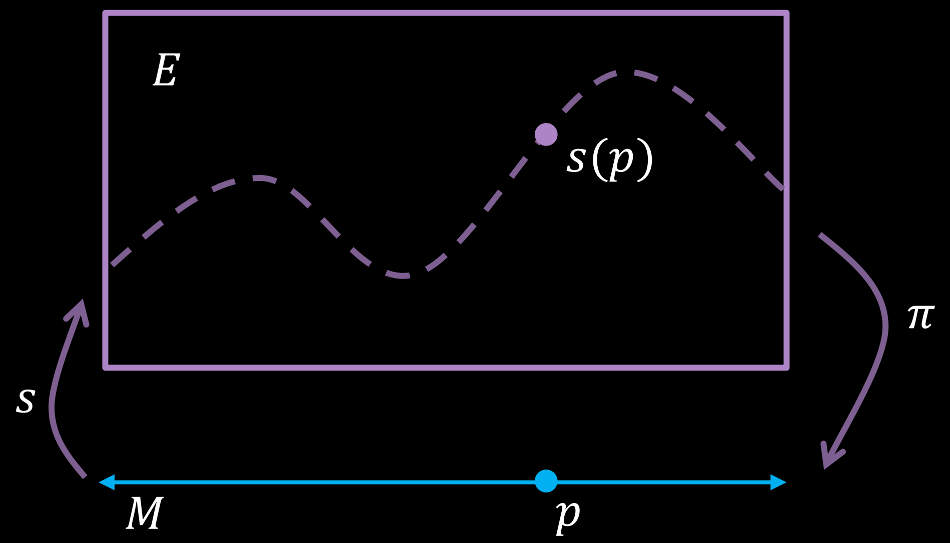


Sections

Fields in physics are often described by ‘sections’ of vector bundles, so when we talk about a general field, we need to talk about sections.

A section of a bundle $\pi: E \to M$ is a function $s: M \to E$ such that for any $p \in M$,

$$s(p) \in E_p$$



Sections

$\mathrm{End}(E)$ denotes the set of all endomorphisms of a vector bundle E

Any section T of $\mathrm{End}(E)$ defines a map from E to itself sending $v \in E_p$ to $T(p)v \in E_p$. So, a section T acts on a section s pointwise, which gives a new section Ts of E

$$(Ts)(p) = T(p)s(p)$$

Therefore, T is a new function

$$T: \Gamma(E) \to \Gamma(E)$$

Where $\Gamma(E)$ is the set of all sections of E

Connections

A connection D on M assigns a function $D_v: \Gamma(E) \rightarrow \Gamma(E)$ to each vector field v on M which satisfies the properties:

$$D_v(\alpha s) = \alpha D_v s$$
$$D_v(s + t) = D_v s + D_v t$$
$$D_v(fs) = v(f)s + f D_v s$$
$$D_{v+w} s = D_v s + D_w s$$
$$D_{fv} s = f D_v s$$

For all $v, s \in \mathrm{Vect}(M), s, t \in \Gamma(E), f \in C^{\infty}(M)$, and all scalars α

We call $D_v s$ the covariant derivative of s in the direction v

Exterior Covariant Derivative

We define the exterior covariant derivative d_D of a section s of E to be the E-valued 1-form d_Ds such that

$$d_D s(v) = D_v s$$

For any vector field v on M.

Note that this is just a generalization of our original exterior derivative

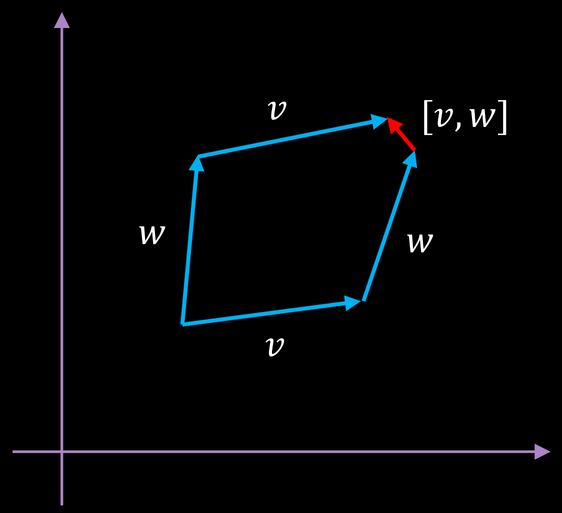
$$df(v) = vf$$

Curvature

The curvature is an operator acting on sections of E that measures the failure of covariant derivatives to commute. For two vector fields v and w on M, the curvature acting on a section s is

$$F(v,w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s$$

The first two terms are $[D_v, D_w]$, which measures their failure to commute and $-D_{[v,w]}s$ measures the effect of a non-vanishing Lie bracket, better shown in the figure



Curvature

$F(v, w)$ is a section of End(E), so when we are working with coordinates on an open set, the 'components'

$$F_{\mu\nu} = F(\partial_\mu, \partial_\nu)$$

are sections of End(E) over the open set. It turns out, we can view the curvature of the connection D on E as an End(E)-valued 2-form,

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

The factor of 1/2 is there to account for the double counting that occurs from $F_{\mu\nu} = -F_{\nu\mu}$ and $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$

The Yang-Mills Equation

Now that we have an analog for the exterior derivative on sections and a generalization for F, we can use a similar analysis as before and defining a Hodge star operator, which leads us to the Yang-Mills equation

$$\star d_D \star F = J$$

This looks very similar to the second equation. In fact, the only difference is that we are are not restricted to any particular type of manifold, as we were earlier.