Connection Problems in Graphs

Jason Rosenhouse
Department of Mathematics and Statistics
James Madison University

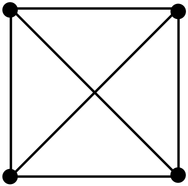
MD-DC-VA Section Meeting
at Virginia State University

April 29, 2023

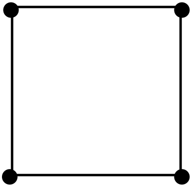
What Are Graphs?

A graph is an ordered pair $G = (V, E)$ where

- V is a set of vertices.
- $E \subset V \times V$ is a set of edges.



K_4



C_4

What Are Graphs?

- We assume that our edges are not directed. Our edges are line segments, not arrows.
- We assume that every edge has two distinct endpoints. There are no loops.
- We assume there is no more than one edge between any pair of vertices. There are no multiple edges.
- A graph is said to be *connected* if for any pair of vertices there is a walk between them.

Graphs of this sort are said to be *simple*. In this talk, all of our graphs are assumed to be simple and connected.

What is Algebraic Graph Theory?

This book is concerned with the use of algebraic techniques in the study of graphs. The aim is to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs.

Norman Biggs
Algebraic Graph Theory

The Adjacency Matrix

Let G be a finite, simple graph on n vertices. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices. Then the *adjacency matrix* $A(G)$ is the $n \times n$ matrix whose entries satisfy:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

The matrix $A(G)$ is real and symmetric, and has trace 0. Since the ordering of the vertices is arbitrary, $A(G)$ is determined only up to a permutation of the rows and columns. Thus, our interest will be in properties of $A(G)$ that are invariant with respect to such permutations.

Examples

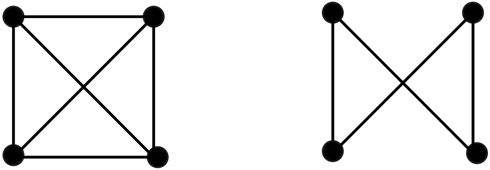


Figure: On the left we have K_4, the complete graph on four vertices. On the right is $K_{2,2}$, the complete bipartite graph on sets of size two.

We have that:

$$A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad A(K_{2,2}) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Two Elementary Results

Theorem

The number of walks of length ℓ in G, from v_i to v_j, is given by the entry in position (i, j) in A^ℓ.

$$A(K_4)^2 = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \qquad A(K_{2,2})^2 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Two Elementary Results

Theorem

Let the characteristic polynomial of $A(G)$ be written as

$$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0.$$

Then the coefficients satisfy:

- $c_{n-1} = 0$.
- $-c_{n-2}$ *is twice the number of edges of* G.
- $-c_{n-3}$ *is twice the number of triangles of* G.

The Eigenvalue Spectrum

Since the eigenvalues of a matrix are invariant under permutations of the rows and columns, they are of special interest in algebraic graph theory. A list of the eigenvalues of the adjacency matrix, with their multiplicities is called the *eigenvalue spectrum* of the graph.

For example:

$$\text{Spec } K_4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \text{Spec } K_{2,2} = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Two More Elementary Results

Theorem

Let G be a k-regular graph. Then k is an eigenvalue of G.

Proof.

If G is k-regular, then $[1, 1, \ldots, 1]^T$ is an eigenvector for $A(G)$. □

Two More Elementary Results

Theorem

Let G be a bipartite graph. Then the eigenvalue spectrum of G is symmetrical around the origin.

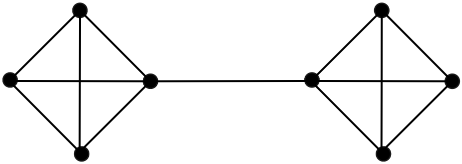
Proof.

(Sketch.) For $K_{2,2}$, we find that $[1, 1, 1, 1]^T$ is an eigenvector with eigenvalue 2, while $[1, 1, -1, -1]^T$ is an eigenvector with eigenvalue -2. This trick always works! □

Competing Concerns

Network design confronts engineers with competing concerns.

- Too few edges make the network vulnerable to attack or mechanical failure. A network shaped like a barbell, for example, would become disconnected by cutting a single edge.



- Too many edges make the network expensive. Generally it is not feasible to design a network in the form of a complete graph.

The Isoperimetric Number

Let G be a graph. Let $S \subset V(G)$. The *edge-boundary* of S, denoted by ∂S, is the set of edges in $E(G)$ with one endpoint in S, and the other in the complement of S.

The *isoperimetric number* of G, denoted by $i(G)$ is then defined by

$$i(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum is taken over all sets S satisfying

$$|S| \leq \frac{1}{2}|V(G)|.$$

Examples

There are special cases where $i(G)$ can be computed explicitly:

1. $i(K_n) = \lceil n/2 \rceil$
2. $i(C_n) = \frac{2}{\lfloor n/2 \rfloor}$
3. $i(P_n) = \frac{1}{\lfloor n/2 \rfloor}$
4. $$i(K_{m,n}) = \begin{cases} mn/m+n & \text{if m,n are even,} \\ (mn+1)/(m+n) & \text{if m,n are odd,} \\ mn/(m+n-1) & \text{if m+n is odd.} \end{cases}$$

In most cases, alas, we must make do with deriving upper and lower bounds for $i(G)$.

The Isoperimetric Number (Continued)

- It is a measure of whether there are bottlenecks in the graph. A low isoperimetric number indicates the presence of a bottleneck. Equivalently, it is a measure of how easy it is to fracture the graph by cutting a small number of edges.
- Related to the first point, in theoretical computer science it is often relevant to study the resiliency of a network.
- Graphs with a large isoperimetric number have strong expansion properties. Roughly, a good expander is a graph that is sparse (in the sense of having a small number of edges relative to the number of vertices), but also resilient (in the sense of being difficult to fracture without cutting a large number of edges.)

The Platonic Graphs

Let Γ be the group $\mathbb{Z}_n \times \mathbb{Z}_n - \{(0,0)\}$. We define the graphs π_n as follows:

- The vertices of π_n are given by $\Gamma/\{\pm 1\}$.
- Vertices (a, b) and (c, d) are connected by an edge iff

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 \pmod{n}.$$

The graphs π_n are called the *Platonic graphs*, because for $n = 3, 4, 5$ they correspond to the 1-skeletons of the tetrahedron, cube, and the dodecahedron.

The Isoperimetric Numbers of the Platonic Graphs

Theorem (Brooks, Perry, Petersen 1993)

Let p be a prime satisfying $p \equiv 1 \pmod 4$. Then

$$i(\pi_p) \leq \frac{(p-1)p}{2(p+1)}.$$

Theorem (Lanphier, Rosenhouse 2004)

Let p be a prime number, and let $r \in \mathbb{Z}^+$. Then we have

$$i(\pi_{p^r}) \leq \begin{cases} \frac{p^r(p-1)}{2(p+1)} & \text{if } p \not\equiv 3 \pmod 4, \\ \frac{p^{2r}-2p^{2r-1}+5p^{2r-2}-4p^{r-1}+4}{2(p^r-2p^{2r-1}-3p^{r-2}+4p^{-1})} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Wheel Graphs

Let C_{n-1} be the cycle on $n-1$ vertices, and let W_n be the graph obtained from C_{n-1} by the addition of a single vertex v, with one edge connecting v to each of the $n-1$ vertices in C_{n-1}.

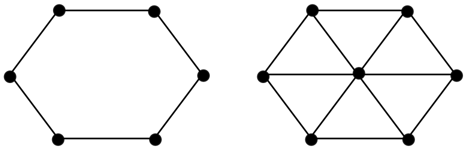


Figure: On the left, the graph C_6. On the right, the graph W_7.

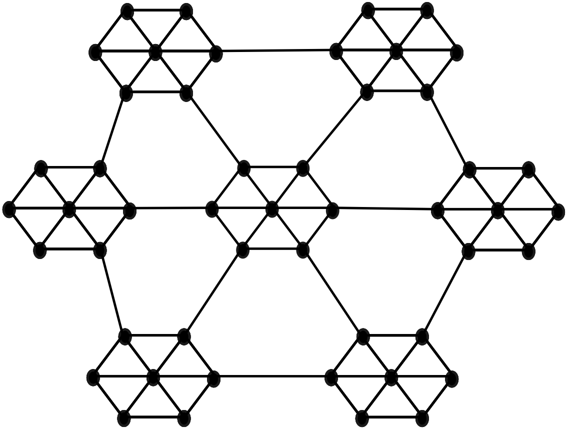
The graphs W_n are referred to as *wheel graphs*.

The Decomposition Theorem

Theorem (Lanphier, Rosenhouse 2004)

- *When p is prime, the graph* π_p *can be partitioned into* $(p-1)/2$ *isomorphic copies of* W_{p+1}*, with* $2p$ *edges joining every pair of wheels. Alternatively,* π_p *is the complete multigraph* $K^{2p}_{(p-1)/2}$*, in which every vertex should be viewed as a wheel.*
- *The graph* π_{p^r} *can be partitioned into two sets A and B so that A is isomorphic to the complete multigraph* $K^{2p^r}_{\Phi(p^r)/2}$*, while vertices in B are joined only to vertices in A, where* Φ *denotes the Euler phi function.*

Illustrating the Decomposition



Cayley Graphs

Let Γ be a group. Let S be a symmetric generating set for Γ. (That is, if $s \in S$ then $s^{-1} \in S$.) Define a graph whose vertices are the elements of Γ, with vertices γ_1 and γ_2 adjacent if there is an $s \in S$ such that $s\gamma_1 = \gamma_2$.

This is called the Cayley graph of Γ with respect to the generating set S. Assuming that S is symmetric ensures that our Cayley graphs are non-directed.

The Platonic Graphs as Cayley Graphs

Let Γ_n denote the group $PSL(2, \mathbb{Z}_n)$. Thus, the entries of Γ_n are

- 2×2 matrices.
- Entries from the integers mod n.
- Determinant $\pm 1 \pmod{n}$.
- Equivalent if they only differ by multiplication by -1.

This group is generated by the elements:

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Platonic Graphs as Cayley Graphs

Thus, the set

$$S = \{U, U^{-1}, V\}$$

is a symmetric generating set for Γ_n.

Define an equivalence relation on the vertices by declaring that $v_1 \sim v_2$ if $v_1 = U^k v_2$ for some positive integer k. The equivalence classes of this relation are circuits of length n.

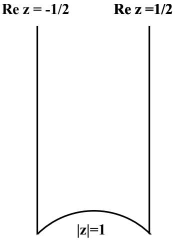
The Platonic graphs π_n are obtained from the Cayley graphs G_n by collapsing each of these circuits to a point.

The Projective Special Linear Groups

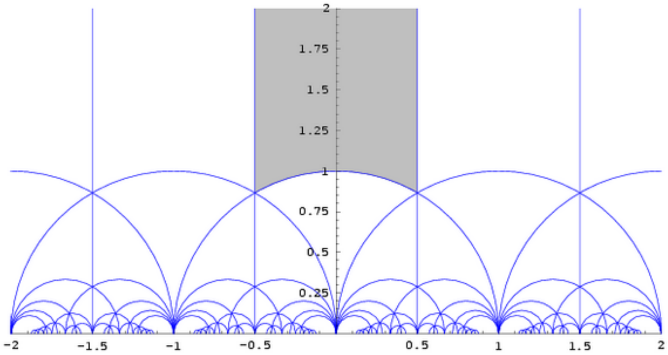
Let $\Gamma = SL(2, \mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1. Then Γ acts on the complex upper half plane via fractional linear transformations:

$$z \mapsto \frac{az + b}{cz + d}.$$

A fundamental domain for this action is given by the standard "modular triangle":



Tiling the Plane With Modular Triangles



Congruence Subgroups

Define the Nth principle congruence subgroup of Γ to be:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

It is possible to construct a simply-connected fundamental domain for the action of $\Gamma(N)$ on the upper half plane by gluing together modular triangles. This fundamental domain can be viewed as a Riemann surface. Using $\mathcal{H}$ to denote the upper half plane, we can denote this surface by $\mathcal{H}/\Gamma(N)$.

The Return of the Platonic Graphs

The surface $\mathcal{H}/\Gamma(N)$ comes "pretriangulated."

We can construct a graph by having one vertex for each modular triangle, with pairs of vertices connected if they share a boundary edge. The resulting graphs are again the Platonic graphs. That is, they are the Cayley graphs of $PSL(2, \mathbb{Z}_n)$ we discussed previously.

The Cheeger Constant

In 1970, Jeff Cheeger defined the following constant, where M is a closed, n-dimensional, Riemannian manifold:

$$h(M) = \inf_N \frac{\text{area}(N)}{\min\left(\text{vol}(A), \text{vol}(B)\right)},$$

where N runs over all $(n-1)$-dimensional hypersurfaces dividing M into two pieces A and B.

If $\lambda_1(M)$ denotes the first eigenvalue of the Laplacian on M, then we have that

$$\lambda_1(M) \geq \frac{1}{4} h(M)^2.$$

The Discrete Cheeger Constant

What we are calling the isoperimetric number is a discrete version of the Cheeger constant introduced by Peter Buser in 1978. His idea was to study the spectral geometry of closed manifolds via the following program:

- Triangulate the manifold.
- Associate a cubic graph to the triangulation.
- Work out the isoperimetric number of the graph.
- Relate the isoperimetric number to the Cheeger constant.

The Isoperimetric Number, Revisited

Every edge in our Cayley graphs represents a boundary edge of a modular triangle in the surface. Moreover, every vertex in the graph represents a modular triangle. It follows that estimates on the isoperimetric number of the Cayley graph lead immediately to estimates on the Cheeger constant of the surface.

More specifically, the edges of a modular triangle have length $\log(3)$, while the modular triangles have area $\pi/3$. It follows that

$$h(\mathcal{H}/\Gamma(N)) \leq \frac{3\log(3)}{\pi} i(G_n).$$

The Picard Modular Group

The Picard modular group is defined as $SL(2, \mathbb{Z}[i])$. As before, we can mod out by a congruence subgroup, leading to the groups $PSL(2, \mathbb{Z}_p[i])$. This group has a standard generating set, and we can consider the Cayley graphs for these groups with respect to this set.

Theorem (Lanphier, Rosenhouse 2004)

The Cayley graphs for the prime quotients of the Picard modular group satisfy a decomposition theorem similar to the one previously described.

The Euclidean Bianchi Groups

Let d be a positive integer. Set $K_d = \mathbb{Q}(\sqrt{-d})$. These are referred to as the Bianchi groups. These groups, and their congruence subgroups, act on complex upper three-space. This leads to certain arithmetic hyperbolic three-manifolds, and there is an analog to Selberg's eigenvalue conjecture for these surfaces. Quotients can be constructed by modding out by a prime ideal, and these quotients have standard generating sets.

For certain choices of d, we have that the ring of integers $\mathcal{O}_d$ is Euclidean. In 2009, we were able to extend our previous results to the Cayley graphs of quotients of the Euclidean Bianchi groups.

Lower Bounds

Finding upper bounds on $i(G)$ is easy. Just pick a set, work out the isoperimetric quotient, and there's your bound.

Lower bounds are harder to come by, but they are possible when the graph is highly connected.

The idea is that when there are a large number of short paths between arbitrary pairs of vertices, we can establish a minimum number of edges that must be cut to separate them.

Lower Bounds for the Platonic Graphs

Using this general approach, we can establish strong lower bounds for the Platonic graphs. Specifically, we have the following:

Theorem (Lanphier, Rosenhouse 2006)

With notation as before, and with $p \equiv 1 \pmod 4$, we have

$$\frac{p^r - \sqrt{p^{2r-2} + 2p^r - 6}}{2} \leq i(\pi_{p^r}) \leq \frac{p^r(p-1)}{2(p+1)}.$$

Note that both bounds approach $p^r/2$ as $p \to \infty$.

Levi Graphs

Levi graphs are bipartite graphs arising from (balanced, incomplete) block designs. The vertices are the points and blocks of the design, with each block adjacent to the points it contains. Levi graphs are highly connected in the sense previously described.

In a 2006 REU project with Christopher Miller and Amber Russell, we were able to derive upper and lower bounds on the isoperimetric numbers of Cayley graphs. In particular, we derived strong bounds for finite projective planes and Hadamard designs.

The End

Thank You!