Fun with L(2,1)-labeling

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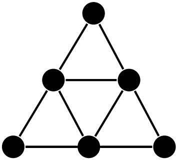
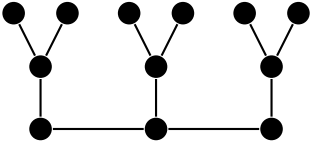
Graphs

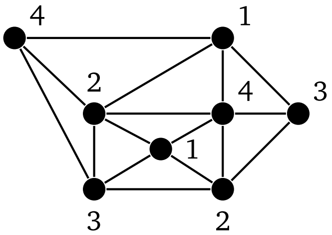
Network of points (vertices) and lines (edges)





Graph coloring

Assign colors (usually numbers) to vertices so that neighbors get different colors. Try to use as few colors as possible.



L(2,1)-labeling

Assign numerical labels to vertices according to the following rules:

1. Neighbors' labels must differ by 2.
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Start with 1 and try to keep the labels as small as possible.

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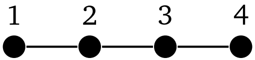

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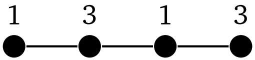

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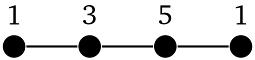

Okay, but not optimal.

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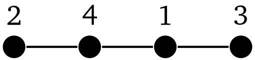

Optimal!

Another example

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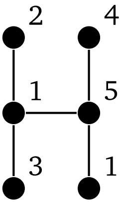

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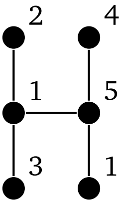

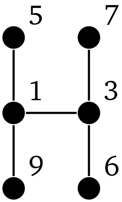

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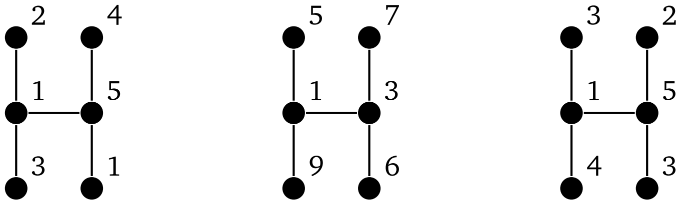

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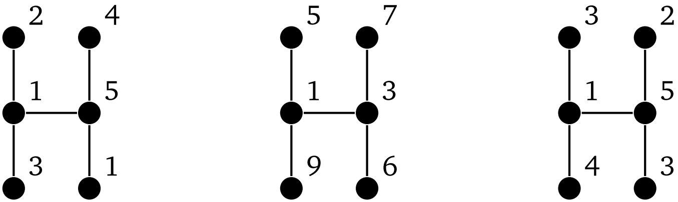
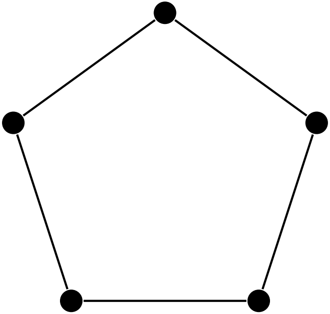


Left graph breaks both rules repeatedly. Middle graph is okay but uses too many colors. Right graph is optimal.

Let’s try another



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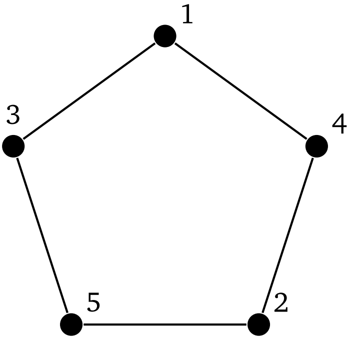

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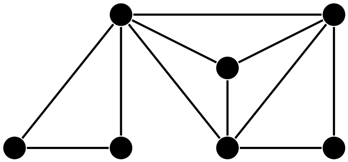

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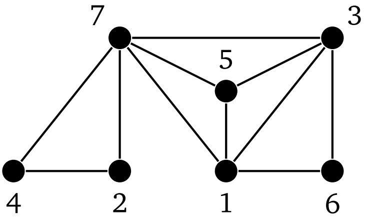
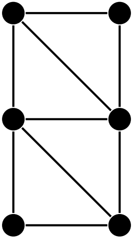


A challenge

Goal: max label 6



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- In the literature, people often talk about the *span*, the difference between the largest and smallest labels.

More info

- $\lambda_{2,1}$ can be computed quickly for various classes of graphs like paths, cycles, complete graphs, etc. These are good exercises.

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- There are many classes of graphs for which $\lambda_{2,1}$ is not known.
- The problem is NP-complete (i.e., interesting).

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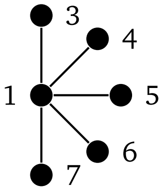

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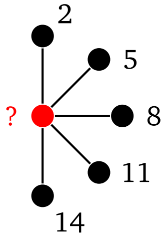

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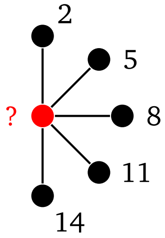

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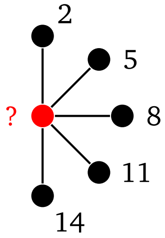

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- In total, we have to avoid at most $3\Delta + \Delta(\Delta - 1) = \Delta^2 + 2\Delta$ labels.

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- Sufficiently large here means 10^{69}.
- They also showed that $\lambda_{2,1} \leq \Delta^2 + c$ for some fixed (but unfortunately very large) c.

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Relatives and generalizations

- Instead of $L(2,1)$, people often look at $L(p,q)$. For example, $L(5,2)$ would mean neighbors' labels must differ by at least 5 and vertices at distance 2 must differ by at least 3.
- Can also look at $L(p,q,r)$, $L(p,q,r,s)$, etc.
- There are also edge versions, list versions, and more.

- MR4556054 Pending Zhang, Xiaoling Optimal $L(2,1,1)$-labelings of caterpillars. *J. Math. Res. Appl.* 43 (2023), no. 2, 150–160. 05C78
  Review PDF | Clipboard | Journal | Article
- MR4568901 Prelim Amanathulla, Sk.; Bera, Biswajit; Pal, Madhumangal; $L(2,1,1)$-labeling of interval graphs. *Int. J. Math. Ind.* 14 (2022), Paper No. 2250003, 8 pp. 05C78
  Review PDF | Clipboard | Journal | Article
- MR4496569 Reviewed Zhu, Haiyang; Zhu, Junlei; Liu, Ying; Wang, Shuling; Huang, Danjun; Miao, Lianying Optimal frequency assignment and planar list $L(2,1)$-labeling. *J. Comb. Optim.* 44 (2022), no. 4, 2748–2761. 05C10 (05C78)
  Review PDF | Clipboard | Journal | Article
- MR4345146 Reviewed Colucci, Lucas; Győri, Ervin On $L(2,1)$-labelings of oriented graphs. *Discuss. Math. Graph Theory* 42 (2022), no. 1, 39–46. 05C78 (05C20)
  Review PDF | Clipboard | Journal | Article
- MR4387595 Reviewed Ma, Dengju; Yao, Shunyu; Dong, Xiaoyuan The $\rm L(3,2,1)$-labeling number of the Cartesian product of a complete graph and a cycle. *Ars Combin.* 154 (2021), 87–100. 05C78
  Review PDF | Clipboard | Journal | Article
- MR4387590 Reviewed Lv, Damei; Lin, Nianfeng $L(2, 1)$-circular-labelings of the edge-path-replacements. *Ars Combin.* 154 (2021), 23–30. 05C78
  Review PDF | Clipboard | Journal | Article
- MR4299390 Reviewed Bandopadhyay, Susobhan; Ghosh, Sasthi C.; Koley, Subhasis Improved bounds on the span of $L(1,2)$-edge labeling of some infinite regular grids. *Graphs and combinatorial optimization: from theory to applications—CTW2020 proceedings,* 53–65, AIRO Springer Ser., 5, *Springer, Cham,* [2021], ©2021. 05C78 (05C12)
  Review PDF | Clipboard | Series | Chapter

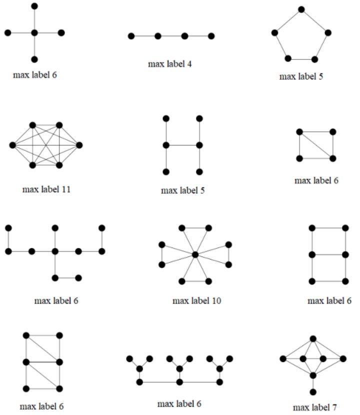
My assignment

Name______________________

Math 211   Assignment 33 (Due Fri 12/2/22)

$L(2,1)$-labeling is a relative of graph coloring. Vertices are assigned integer labels according to the following rules:

- The labels for neighbors must differ by at least 2.
- If a vertex is two steps away from another (i.e., to get to it, you have to pass through one exactly other vertex), then their labels need to be different.
- If vertices are more than two steps away from each other, then it is okay for them to have the same label.
- The "max label" listed under each graph is the largest number you are allowed to use. That label should show up somewhere in the graph. The smallest label used should always be 1. It's possible that not all the numbers between 1 and the max will be needed.



Comments

- I've used it for several years. Students in a non-majors course have told me they found it
  - Fun
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- It's not something they have ever seen before.
- It's also a good problem for undergraduate research since it doesn't require a lot of background to get started on, and there are myriad things to work on.

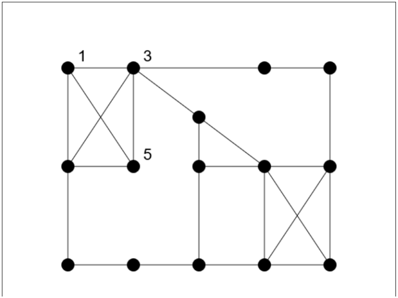
An app

- I find $L(2,1)$-labelling works well as a puzzle. It's easy to make mistakes, though. I built a simple app that checks your work.

  `https://www.brianheinold.net/L21.html`

- You can view source on the page to see the code. Adding new graphs is quick.

App



Board number: 6

Goal: 7

Clear Previous Next Random

Rules:

1. Neighbors' labels must differ by 2.
2. Vertices at distance 2 from each other must get different labels.
3. The smallest label is 1 and the largest label shouldn't be more than the goal.

Thank you!

A few selected references:

T. Calamoneri. The $L(h,k)$-Labeling Problem: A Survey, (2014)
http://www.dsi.uniroma1.it/calamo/PDF-FILES/survey.pdf

J.R. Griggs and R.K. Yeh, Labeling graphs with a condition at distance two, SIAM J. Discr. Math. 11 (1992), 585–595.

Havet, F., Reed, B. and Sereni, J.-S. (2008) $L(2,1)$-Labeling of graphs. Proceedings of the ACM-SIAM Symposium on Discrete Algorithm (SODA '08), San Francisco, California, 20-22 January, 621–630.