Prime Factors and Divisibility of Sums of Powers of Fibonacci Numbers

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Introduction

If a pair of rabbits is placed in an enclosed area, how many pairs of rabbits will there be after a year if we have the following assumptions:

- Every month a pair of rabbits produces another pair
- Rabbits begin to bear young two months after their birth and
- None of the rabbits die

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$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

Research Questions

- Are there patterns in the prime factors of the sum $F_n^2 + F_{n-2}^2$ for all $n \geq 2$?
- Are there different patterns in the prime factors of the sum $F_n^3 + F_{n-2}^3$ for all $n \geq 2$?

Preliminaries

The Recursive Definition:

$$F_{n+2} = F_{n+1} + F_n, \text{ for } n \in \mathbb{N} \text{ with } F_1 = F_2 = 1$$

Preliminaries

Modular Arithmetic

Definition: Let n be a positive integer and let a and b be any integers. We say that a is *congruent* to b mod n written, $a \equiv b \bmod n$, if a and b have the same remainder when divided by n.

Properties:

$$(a \bmod n) + (b \bmod n) \equiv (a + b) \bmod n$$

$$(a \bmod n)(b \bmod n) \equiv (ab) \bmod n$$

Preliminaries

Table: F_n under mod 2

n	1	2	3	4	5	6	7	8	9	10	11
F_n	1	1	2	3	5	8	13	21	34	55	89
F_n mod 2	1	1	0	1	1	0	1	1	0	1	1
F_n^2 mod 2	1	1	0	1	1	0	1	1	0	1	1

Preliminaries

Definition: A *divisibility sequence* is an integer sequence, $\{a_n\}$, indexed by positive integers n, such that if m divides n then a_m divides a_n.

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Proposition

1. *$F_3 = 2$ divides every third Fibonacci number.*
2. *$F_4 = 3$ divides every fourth Fibonacci number.*
3. *$F_5 = 5$ divides every fifth Fibonacci number.*

Results

Lemma

For all $n \in \mathbb{N}_0$,

$$F_{3n+4}^2 + F_{3n+2}^2 \text{ is even.}$$

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$$\begin{aligned} F_{3n+4}^2 + F_{3n+2}^2 &= (F_{3n+3} + F_{3n+2})^2 + F_{3n+2}^2 \\ &= F_{3n+3}^2 + 2F_{3n+3}F_{3n+2} + 2F_{3n+2}^2 \end{aligned}$$

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For all $n \in \mathbb{N}_0$,

a. $F_{5n+3}^2 + F_{5n+1}^2$ *is a multiple of 5.*

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F_n	1	1	2	3	5	8	13	21	34	55	89
F_n mod 5	1	1	2	3	0	3	3	1	4	0	4
F_n^2 mod 5	1	1	4	4	0	4	4	1	1	0	1

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For all $n \in \mathbb{N}_0$, $F_{5n+3}^2 + F_{5n+1}^2$ is a multiple of 5.

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&= 3F_{5n+1}^2 + F_{5n}^2 + 2F_{5n+1}F_{5n} + 2F_{5n+2}F_{5n+1} \\
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For all $n \in \mathbb{N}_0$, $F_{5n+3}^2 + F_{5n+1}^2$ and $F_{5n+4}^2 + F_{5n+2}^2$ are multiples of 5.

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$$\begin{aligned} F_{5n+3}^2 + F_{5n+1}^2 &= (F_{5n+2} + F_{5n+1})^2 + F_{5n+1}^2 \\ &= F_{5n+2}^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2 \\ &= (F_{5n+1} + F_{5n})^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2 \\ &= 3F_{5n+1}^2 + F_{5n}^2 + 2F_{5n+1}F_{5n} + 2F_{5n+2}F_{5n+1} \\ &= 5F_{5n+1}^2 + F_{5n}^2 + 4F_{5n+1}F_{5n} \end{aligned}$$

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Table: F_n under mod 5

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F_n	1	1	2	3	0	3	3	1	4	0	4	4	3	2
F_n^2	1	1	4	4	0	4	4	1	1	0	1	1	4	4

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F_n	1	1	2	3	0	3	3	1	4	0	4	4	3	2
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For all $n \geq 2$,

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For all $n \geq 2$, $F_n^2 + F_{n-2}^2$ will never have a factor of 3.

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a. $\mathbf{F_n^2}$ **and** $\mathbf{F_{n-2}^2}$ **both have a factor of 3; that is** $\mathbf{F_n^2 \equiv F_{n-2}^2 \equiv 0 \bmod 3}$

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F_n^2 mod 3	1	1	1	0	1	1	1	0	1	1	1

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Lemma

For all $n \geq 2$, $F_n^2 + F_{n-2}^2$ will never have a factor of 3.

Proof.

b. $\mathbf{F_n^2 \equiv 1 \bmod 3}$ **and** $\mathbf{F_{n-2}^2 \equiv 2 \bmod 3}$ **or vice versa.**

n	1	2	3	4	5	6	7	8	9	10	11
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$F_n \bmod 3$	1	1	2	0	2	2	1	0	1	1	2
$F_n^2 \bmod 3$	1	1	1	0	1	1	1	0	1	1	1

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Conclusions and Further Work

Lemma

For all $n \in \mathbb{N}_0$, $F_{4n+3}^3 + F_{4n+1}^3$ *is divisible by 3.*

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For all $n \in \mathbb{N}_0$, $F_{4n+3}^3 + F_{4n+1}^3$ is divisible by 3.

Conjecture

For all $n \in \mathbb{N}_0$, $F_{14n+5}^2 + F_{14n+3}^2$ is divisible by 29.

$$F_{14n+11}^2 + F_{14n+9}^2 \text{ is divisible by 29.}$$

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For all $n \in \mathbb{N}_0$, $F_{4n+3}^3 + F_{4n+1}^3$ *is divisible by 3.*

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$F_{14n+11}^2 + F_{14n+9}^2$ *is divisible by 29.*

Conjecture

For all $n \in \mathbb{N}_0$, $L_{3n+5}^2 + L_{3n+3}^2$ *is divisible by 2.*

$L_{n+3}^3 + L_{n+1}^3$ *is divisible by 5.*

Questions

Thank You! Questions?

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