Prime Factors and Divisibility of Sums of Powers of Fibonacci Numbers
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Spirit Karcher

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If a pair of rabbits is placed in an enclosed area, how many pairs of rabbits will there be after a year if we have the following assumptions:

- Every month a pair of rabbits produces another pair
- Rabbits begin to bear young two months after their birth and
- None of the rabbits die
If a pair of rabbits is placed in an enclosed area, how many pairs of rabbits will there be after a year if we have the following assumptions:

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- Rabbits begin to bear young two months after their birth and
- None of the rabbits die

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
Research Questions

- Are there patterns in the prime factors of the sum $F_n^2 + F_{n-2}^2$ for all $n \geq 2$?

- Are there different patterns in the prime factors of the sum $F_n^3 + F_{n-2}^3$ for all $n \geq 2$?
The Recursive Definition:

\[ F_{n+2} = F_{n+1} + F_n, \text{ for } n \in \mathbb{N} \text{ with } F_1 = F_2 = 1 \]
Modular Arithmetic

**Definition:** Let $n$ be a positive integer and let $a$ and $b$ be any integers. We say that $a$ is *congruent* to $b$ mod $n$ written, $a \equiv b \mod n$, if $a$ and $b$ have the same remainder when divided by $n$.

**Properties:**

$$(a \mod n) + (b \mod n) \equiv (a + b) \mod n$$

$$(a \mod n)(b \mod n) \equiv (ab) \mod n$$
### Preliminaries

Table: $F_n$ under mod 2

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
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<tr>
<td>$F_n$</td>
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<td>13</td>
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<tr>
<td>$F_n$ mod 2</td>
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Preliminaries

Definition: A divisibility sequence is an integer sequence, \( \{a_n\} \), indexed by positive integers \( n \), such that if \( m \) divides \( n \) then \( a_m \) divides \( a_n \).

Example: \( F_3 = 2 \) so when, \( n = 3k \), \( 2 \) divides \( F_3^k \).

Proposition 1. \( F_3 = 2 \) divides every third Fibonacci number.

2. \( F_4 = 3 \) divides every fourth Fibonacci number.

3. \( F_5 = 5 \) divides every fifth Fibonacci number.
Definition: A divisibility sequence is an integer sequence, \( \{a_n\} \), indexed by positive integers \( n \), such that if \( m \) divides \( n \) then \( a_m \) divides \( a_n \).

Example: \( F_3 = 2 \) so when, \( n = 3k \), 2 divides \( F_{3k} \).
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**Proposition**

1. \( F_3 = 2 \) divides every third Fibonacci number.
2. \( F_4 = 3 \) divides every fourth Fibonacci number.
3. \( F_5 = 5 \) divides every fifth Fibonacci number.
Results

**Lemma**

For all $n \in \mathbb{N}_0$, 

$$F_{3n+4}^2 + F_{3n+2}^2 \text{ is even.}$$
Results

Lemma

For all $n \in \mathbb{N}_0$, $F_{3n+4}^2 + F_{3n+2}^2$ is even.

Proof.

$F_{3n+4}^2 + F_{3n+2}^2 =$
Lemma

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Proof.

$$F_{3n+4}^2 + F_{3n+2}^2 = (F_{3n+3} + F_{3n+2})^2 + F_{3n+2}^2$$

$$= F_{3n+4}^2$$
Results

Lemma

For all \( n \in \mathbb{N}_0 \),

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Proof.

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F_{3n+4}^2 + F_{3n+2}^2 = (F_{3n+3} + F_{3n+2})^2 + F_{3n+2}^2 \\
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$$= F_{3n+3}^2 + 2F_{3n+3}F_{3n+2} + 2F_{3n+2}^2$$
Results

Lemma

For all $n \in \mathbb{N}_0$,

\begin{align*}
\text{a. } & F_{5n+3}^2 + F_{5n+1}^2 \text{ is a multiple of 5.} \\
\text{b. } & F_{5n+4}^2 + F_{5n+2}^2 \text{ is a multiple of 5.}
\end{align*}
Results

Lemma

For all $n \in \mathbb{N}_0$,

a. $F_{5n+3}^2 + F_{5n+1}^2$ is a multiple of 5.

b. $F_{5n+4}^2 + F_{5n+2}^2$ is a multiple of 5.

<table>
<thead>
<tr>
<th>$F_n$</th>
<th>1</th>
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<tr>
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<td>$F_n^2 \text{ mod } 5$</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Results

Lemma

For all $n \in \mathbb{N}_0$, $F_{5n+3}^2 + F_{5n+1}^2$ is a multiple of 5.

Proof.

\[
F_{5n+3}^2 + F_{5n+1}^2 = (F_{5n+2} + F_{5n+1})^2 + F_{5n+1}^2
= F_{5n+3}^2
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Results

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= F_{5n+2}^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1} \\
= (F_{5n+1} + F_{5n})^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1} \\
= F_{5n+2}^2
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For all \( n \in \mathbb{N}_0 \), \( F_{5n+3}^2 + F_{5n+1}^2 \) is a multiple of 5.

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= F_{5n+2}^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2 \\
= (F_{5n+1} + F_{5n})^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2 \\
= 3F_{5n+1}^2 + F_{5n}^2 + 2F_{5n+1}F_{5n} + 2F_{5n+2}F_{5n+1}
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Lemma

For all \( n \in \mathbb{N}_0 \), \( F_{5n+3}^2 + F_{5n+1}^2 \) is a multiple of 5.

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= (F_{5n+1} + F_{5n})^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2 \\
= 3F_{5n+1}^2 + F_{5n}^2 + 2F_{5n+1}F_{5n} + 2F_{5n+1} \left( F_{5n+1} + F_{5n} \right) \\
= F_{5n+2}
\]
Results

Lemma

For all \( n \in \mathbb{N}_0 \), \( F_{5n+3}^2 + F_{5n+1}^2 \) is a multiple of 5.

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F_{5n+3}^2 + F_{5n+1}^2 = (F_{5n+2} + F_{5n+1})^2 + F_{5n+1}^2 \\
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= 3F_{5n+1}^2 + F_{5n}^2 + 2F_{5n+1}F_{5n} + 2F_{5n+2}F_{5n+1} \\
= 5F_{5n+1}^2 + F_{5n}^2 + 4F_{5n+1}F_{5n}
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Lemma

For all $n \in \mathbb{N}_0$, $F_{5n+3}^2 + F_{5n+1}^2$ is a multiple of 5.

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F_{5n+3}^2 + F_{5n+1}^2 = (F_{5n+2} + F_{5n+1})^2 + F_{5n+1}^2
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= 5F_{5n+1}^2 + F_{5n}^2 + 4F_{5n+1}F_{5n}
\]
Results

Lemma

For all $n \in \mathbb{N}_0$, $F_{5n+3}^2 + F_{5n+1}^2$ and $F_{5n+4}^2 + F_{5n+2}^2$ are multiples of 5.

Proof.

$$F_{5n+3}^2 + F_{5n+1}^2 = (F_{5n+2} + F_{5n+1})^2 + F_{5n+1}^2$$
$$= F_{5n+2}^2 + 2F_{5n+2}F_{5n+1} + 2F_{5n+1}^2$$
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**Table: $F_n$ under mod 5**

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<th>89</th>
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<th>233</th>
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</table>
## Results

**Table: \( F_n \) under mod 5**

<table>
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<tr>
<th>( F_n )</th>
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Results

Lemma

For all \( n \geq 2 \),

\[ F_n^2 + F_{n-2}^2 \] will never have a factor of 3.

Proof.
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\[ F_n^2 + F_{n-2}^2 \] will never have a factor of 3.

Proof.

a. \( F_n^2 \) and \( F_{n-2}^2 \) both have a factor of 3; that is
\[ F_n^2 \equiv F_{n-2}^2 \equiv 0 \mod 3, \] or
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b. \( F_n^2 \equiv 1 \mod 3 \) and \( F_{n-2}^2 \equiv 2 \mod 3 \) or vice versa.
Results

Lemma

For all $n \geq 2$, $F_n^2 + F_{n-2}^2$ will never have a factor of 3.

Proof.

a. $F_n^2$ and $F_{n-2}^2$ both have a factor of 3; that is $F_n^2 \equiv F_{n-2}^2 \equiv 0 \mod 3$
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Lemma

For all \( n \geq 2 \), \( F_n^2 + F_{n-2}^2 \) will never have a factor of 3.

Proof.

b. \( F_n^2 \equiv 1 \mod 3 \) and \( F_{n-2}^2 \equiv 2 \mod 3 \) or vice versa.
Results

Lemma

For all \( n \geq 2 \),

\[ F_n^2 + F_{n-2}^2 \text{ will never have a factor of 3.} \]

Proof.

a. \( F_n^2 \) and \( F_{n-2}^2 \) both have a factor of 3; that is

\[ F_n^2 \equiv F_{n-2}^2 \equiv 0 \mod 3, \text{ or} \]

b. \( F_n^2 \equiv 1 \mod 3 \) and \( F_{n-2}^2 \equiv 2 \mod 3 \) or vice versa.
Conclusions and Further Work

Lemma

For all \( n \in \mathbb{N}_0 \), \( F_{4n+3}^3 + F_{4n+1}^3 \) is divisible by 3.
Conclusions and Further Work

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For all \( n \in \mathbb{N}_0 \), \( F_{4n+3}^3 + F_{4n+1}^3 \) is divisible by 3.

Conjecture

For all \( n \in \mathbb{N}_0 \), \( F_{14n+5}^2 + F_{14n+3}^2 \) is divisible by 29.

\( F_{14n+11}^2 + F_{14n+9}^2 \) is divisible by 29.
Conclusions and Further Work

**Lemma**

For all $n \in \mathbb{N}_0$, $F_{4n+3}^3 + F_{4n+1}^3$ is divisible by 3.

**Conjecture**

For all $n \in \mathbb{N}_0$, $F_{14n+5}^2 + F_{14n+3}^2$ is divisible by 29.

$F_{14n+11}^2 + F_{14n+9}^2$ is divisible by 29.

**Conjecture**

For all $n \in \mathbb{N}_0$, $L_{3n+5}^2 + L_{3n+3}^2$ is divisible by 2.

$L_{n+3}^3 + L_{n+1}^3$ is divisible by 5.
Questions

Thank You! Questions?
References


