Ten Mathematical Landmarks, 1967–2017

MD-DC-VA Spring Section Meeting
Frostburg State University
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Nine, Eight, Seven, and Six

Nine: Chaos, Fractals, and the Resurgence of Dynamical Systems

Eight: The Explosion of Discrete Mathematics

Seven: The Technology Revolution

Six: The Classification of Finite Simple Groups

Five: The Four-Color Theorem

1852-1976

- 1852: Francis Guthrie's conjecture: Four colors are sufficient to color any map on the plane in such a way that countries sharing a boundary edge (not merely a corner) must be colored differently.
- 1878: Arthur Cayley addresses the London Math Society, asks if the Four-Color Theorem has been proved.
- 1879: A. B. Kempe publishes a proof.
- 1890: P. J. Heawood notes that Kempe's proof has a serious gap in it, uses same method to prove the Five-Color Theorem for planar maps. The Four-Color Conjecture remains open.
- 1891: Heawood makes a conjecture about coloring maps on the surfaces of many-holed tori.
- 1900-1950's: Many attempts to prove the 4CC (Franklin, George Birkhoff, many others) give a jump-start to a certain branch of topology called *Graph Theory*. The latter becomes very popular.

Five: The Four-Color Theorem (4CT)

The first proof

- By the early 1960s, the 4CC was proved for all maps with at most 38 regions.
- In 1969, H. Heesch developed the two main ingredients needed for the ultimate proof, namely reducibility and discharging. He also introduced the idea of an unavoidable configuration.
- In the early 1970s, Ken Appel and Wolfgang Haken put the pieces together, used Heesch's ideas and 1200 hours of CPU time on the University of Illinois' ILLIAC computer ...
- ... and announced a proof in 1976.
- Great was the uproar in the mathematics community. Appel and Haken's proof was later refined and simplified, and we now call it the Four-Color Theorem.

Four: Terence Tao, the Ramanujan of today

Life

- 1975: b. Australia. A child prodigy – and how!
- 1986-88: IMO, bronze, silver, & gold medals.
- 1996: PhD, Princeton; joined math faculty at UCLA.
- 1999: promoted to full professor (youngest ever at UCLA).

Achievements

- 2006: Fields Medal, MacArthur "Genius" Fellowship
- 2007: Fellow of the Royal Society
- Over 300 research papers, 17 books
- Encourages collaboration – over 70 co-authors
- 2015: Breakthrough Prize in Mathematics

... and on top of all that:

- A modest, unassuming, nice guy.

Fields Medalist Timothy Gowers on Terence Tao

"Tao's mathematical knowledge has an extraordinary combination of breadth and depth: he can write confidently and authoritatively on topics as diverse as partial differential equations, analytic number theory, the geometry of 3-manifolds, group theory, quantum mechanics, probability, combinatorics, image processing, functional analysis, and many others. How he does all this while writing papers at a prodigious rate, is a complete mystery. Supposedly, Hilbert was the last person to know all of mathematics, but it is not easy to find gaps in Tao's knowledge, and if you do then you may well find that the gaps have been filled a year later."

Twin Primes

- Twin primes are two primes, such as $\{3, 5\}$, $\{107, 109\}$, and $\{2003663623 \times 2^{195000} \pm 1\}$, that differ by 2. The name dates from the late nineteenth century.
- Twin Prime conjecture: There are infinitely many pairs of twin primes.
- 1915: Viggo Brun proves that $S = \displaystyle\sum_{p:\, p+2 \text{ is prime}} \frac{1}{p}$ converges.

Three: Bounded Gaps and Yitang Zhang

What is a bounded gap?

- The Twin-Prime Conjecture states that the difference between consecutive primes is bounded by 2 infinitely often.
- A bounded gap theorem with a bound of K states that the difference between consecutive primes is bounded by K infinitely often.

Yitang Zhang's amazing achievement

- In 2013, Yitang Zhang announced a proof of a bounded gap theorem with $K = 70,000,000$. He was 58 years old at the time.
- He had been an untenured Instructor of Mathematics at the University of New Hampshire for 13 years. The following year, he was given a tenured Professorship in the Math Department at UNH, followed by a MacArthur Foundation Grant (a so-called "genius grant"), and a tenured professorship at UC Santa Barbara.

Two: The Poincaré Conjecture

The conjecture (1904)

- A(n) *n-manifold* is a space (i.e. a set of points and neighborhoods) that locally looks like (n-dimensional) Euclidean space $\mathbb{R}^(n)$.
- The n-sphere is the boundary of the $(n+1)$-ball. (Thus, the 2-sphere is the (hollow) surface of a (solid) 3-dimensional ball. The 1-sphere is a circle. The 0-sphere is a pair of points.)
- In 1904, Henri Poincaré (1854-1912) made a general conjecture about n-manifolds.
- We are interested in the case $n = 3$, where it says that the 3-sphere is the only possible kind of bounded 3-manifold that has no holes.

Two: The Poincaré Conjecture

Grigori Perelman and Poincaré 3

- In 2002-3, Perelman posted three preprints on the Math arXiv; together, these amounted to a proof of Bill Thurston's Geometrization conjecture, an overarching theorem that produced Poincaré 3 as a corollary.
- Much uproar followed.
- In 2006, Perelman was awarded a Fields medal by the International Mathematical Congress. He refused to accept it.
- In 2010, Perelman was awarded a $1,000,000 prize by the Clay Math Institute for solving one of the seven Millennial Problems. He refused that one, too.
- In both cases, Perelman said that he could not have constructed his proofs without the work of Richard Hamilton in the 1980s showing the way.

One: TeX

A writer gets mad at a typesetting system

- Donald Knuth (b. 1938) publishes the first two volumes of *The Art of Computer Programming*, in 1968 – it is another landmark event from the past fifty years.
- 1977: Knuth gets galley proofs of the third volume. They are awful. Knuth is plenty mad.
- So he designs his own typesetting system, called TeX.
- In the 1980s, he gives the TeX copyright to the AMS.

The universal communicator of mathematics

Later versions, including LaTeX, have revolutionized how we communicate mathematics. It is easy to learn and the raw TeX has become a way to linearize mathematics – thus, it can be communicated via email. The only hard part of mathematics now is the fun part: research.

"But if it wasn't hard, anyone could do it."

Zero: Fermat's Last Theorem (FLT)

From Fermat to Germain

- 1637: Fermat's marginal note in Diophantus' *Arithmetica*, near Pythagorean Triples:
- "However, a sum of two cubes cannot be a cube . . . and in general no higher power can be a sum of two numbers of the same power.
- "I have discovered a truly marvelous proof, but the margin is not large enough to contain it." That is:
- FLT: If $n > 2$, then the equation $x^n + y^n = z^n$ has no solutions in nonzero integers x, y, and z.
- In 1640, Fermat proves FLT for $n = 4$. This is the first proof by infinite descent.

1730–1830

- Euler, Legendre, Gauss, Dirichlet, and Germain: FLT is true for $n < 100$, with three exceptions.

Zero: Fermat's Last Theorem (FLT)

Algebraic Number Theory and the computer

- 1848: Kummer, unique factorization, regular and irregular primes
- The birth of algebraic number theory: ideal numbers
- The birth of abstract algebra: ideals, rings, domains, fields, etc.
- FLT shown true for regular primes and for all $n < 4500$.
- By the 1970s, FLT known for n up in the millions.
- No substantial change in the approach.

The 1980s

- Gerhard Frei gives a talk, suggests a new approach using some mathematical objects that have a whole lot of structure:elliptic curves
- Andrew Wiles, in attendance, says something like "Now that's the first really great idea I've heard of." In short:
- "They've Been Digging In The Wrong Place."

Zero: Fermat's Last Theorem (FLT)

Wiles' Announcement, June 1993

- " ... and that proves Fermat's Last Theorem."
- "I think I'll stop there." Thunderous applause.
- Within six months, a flaw is found in Wiles' argument.

... and after 350 years:

- Richard Taylor to the rescue.
- September 1994: Wiles sees how to salvage the proof.
- September 1995: Two papers appear, one by Taylor and Wiles, the other by Wiles alone. The upshot:
- Fermat's Last Theorem is finally a Theorem!

THANK YOU!