# Polynomial Equations and Tangents

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A presentation to the Mathematical Association of America

MD-DC-VA Section Meeting

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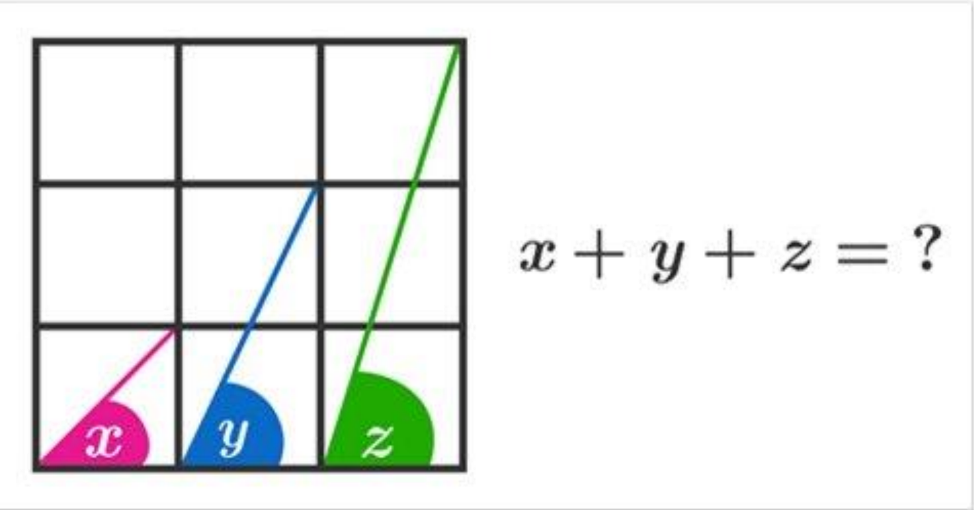
# Brilliant.org Puzzle




- Problem appeared in a Facebook post this past winter
- What is the sum of x, y, and z?
- It is $\arctan(1) + \arctan(2) + \arctan(3)$
- But how do you evaluate that?

# Example 2

- What is $\tan(\arctan(2\cos(2\pi/9)+\arctan(2\cos(8\pi/9)+\arctan(2\cos(14\pi/9))$ ?

# Example 2

- What is $\tan(\arctan(2\cos(2\pi/9)) + \arctan(2\cos(8\pi/9)) + \arctan(2\cos(14\pi/9)))$ ?
- $\tan(\arctan(2\cos(2\pi/9)) + \arctan(2\cos(8\pi/9)) + \arctan(2\cos(14\pi/9))) = \frac{1}{4}$

# Example 3

College Mathematical Journal, 2014 November, p. 357

**Proof Without Words: An Arctangent Identity**

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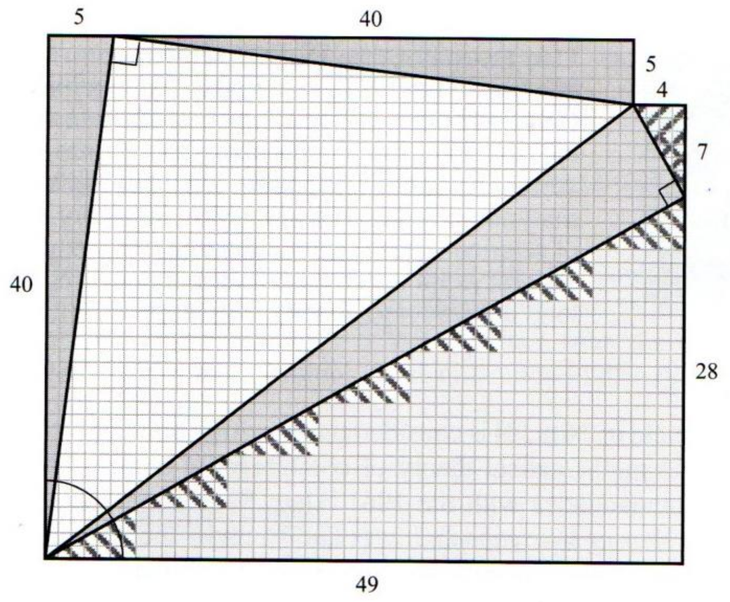



$$\arctan\frac{1}{8} + \frac{\pi}{4} + \arctan\frac{1}{7} + \arctan\frac{4}{7} = \frac{\pi}{2}.$$

# Tangents

- tan(arctan(1)) = 1, tan(arctan(2))=2, tan(arctan(3))=3
- This suggests we work with sums of tangents.

$$\tan(A+B) = \tan(A) + \tan(B)$$

# Tangents

- tan(arctan(1)) = 1, tan(arctan(2))=2, tan(arctan(3))=3
- This suggests we work with sums of tangents.

$$\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

# Sum of Tangents

- High school trigonometry, but we want tan(A+B+C)

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$\tan(A+B) = \frac{\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)} + \frac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)} - \frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}$$

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

# Tangents of sums

- Note that the tangent formula is expressed only with tangents, not with other trig functions
- This allows us to evaluate tan(A+B+C) as tan((A+B)+C) and use the tangent formula twice.

# Tangent of sum of three angles

$$\tan(A+B+C) = \tan((A+B)+C)$$

$$= \frac{\tan(A+B)+\tan(C)}{1-\tan(A+B)\tan(C)}$$

$$= \frac{\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}+\tan(C)}{1-\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}\tan(C)}$$

$$= \frac{\tan(A)+\tan(B)+(1-\tan(A)\tan(B))\tan(C)}{1-\tan(A)\tan(B)-(\tan(A)+\tan(B))\tan(C)}$$

$$\tan(A+B+C) = \frac{\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C)}{1-\tan(A)\tan(B)-\tan(A)\tan(C)-\tan(B)\tan(C)}$$

# Symmetric Polynomials

$$\tan(A+B+C)=\frac{\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C)}{1-\tan(A)\tan(B)-\tan(A)\tan(C)-\tan(B)\tan(C)}$$

- The terms in the formula are symmetric in tan(A), tan(B) and tan(C)
- They are symmetric polynomials, as are coefficients of polynomials
- This suggests finding a polynomial equation which has tan(A), tan(B) and tan(C) as roots; let $r_1$, $r_2$, and $r_3$ be tan(A), tan(B) and tan(C ) respectively

$$\tan(A+B+C)=\frac{r_1+r_2+r_3-r_1r_2r_3}{1-(r_1r_2+r_1r_3+r_2r_3)}$$

$$=\frac{-a+c}{1-b}=-\frac{a-c}{1-b}$$

$$x^3+ax^2+bx+c=0$$

# Tangent of four angles

$$\tan(A+B+C+D)=\frac{\tan(A)+\tan(B)+\tan(C)+\tan(D)-\tan(A)\tan(B)\tan(C)+\tan(A)\tan(B)\tan(D)+\tan(A)\tan(C)\tan(D)+\tan(B)\tan(C)\tan(D)}{1-(\tan(A)\tan(B)+\tan(A)\tan(C)+\tan(A)\tan(D)+\tan(B)\tan(C)+\tan(B)\tan(D)+\tan(C)\tan(D))+\tan(A)\tan(B)\tan(C)\tan(D)}$$

$$=-\frac{a_1-a_3}{1-a_2+a_4}$$

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0$$

- Note the minus sign in in front of Line 2. That's because $a_k$, where k is odd, is the negative of the sum of products in the term.
- This seems to confirm the pattern. That suggests a theorem, but first of all some new notation.

# Some notation

- Define

$$[n_1 n_2 \cdots n_j] = \sum r_{k_1}{}^{n_1} r_{k_2}{}^{n_2} \cdots r_{k_j}{}^{n_j}$$

- Where the sum is taken over all *j*-subsets of roots. If the number of roots is less than *j*, this is defined to be 0.
- Example

$$\begin{aligned} &[31] \\ &= 0 \\ &= a^3 b + ab^3 \\ &= a^3 b + a^3 c + b^3 a + b^3 c + c^3 a + c^3 b \\ &\cdots \end{aligned}$$

- Where a, b, … are the roots of a polynomial equation

# Bracket and Brace Notation

- For example,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+ac+bc)$$

$$[1]^2 = [2] + 2[11]$$

- The coefficients of a polynomial equation are $(-1)^k[11\ldots1]$ for *k* 1's, where the string of *k* 1's corresponds to $a_k$ .
- To avoid writing strings of 1's, make another definition

$$\{n\} = \underbrace{[11\ldots1]}_{\text{n 1's}}$$

- If the number of roots < *n*, then $\{n\}=0$.

# Bracket and brace notation

- Then for any polynomial f(x),

$$f(x) = x^n + a_1 x^{n-1} + \cdots a_n = x^n - \{1\} x^{n-1} + \cdots + (-1)^{n-1}\{n-1\}x + (-1)^n \{n\}$$

- This is much cleaner than complicated summation symbols or long strings of monomials.
- Note this is independent of the particular root values or even of the number of roots (degree of equation).
- Note that for all *j*, $a_j = (-1)^j \{j\}$
- And also that I number coefficients upward instead of downward in the conventional form.

# Bracket and Brace Notation

- When it is necessary to include the number of roots, write

$$[n_1 n_2 \cdots n_k]_p$$
$$\{n\}_p$$

- where $p$ is the number of roots.
- This is zero if $p<k$ and $p<n$, respectively.

# Polynomial-tangent theorem

- Suppose that the roots of an equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} \cdots a_n = 0$
- are $\tan(r_1), \cdots \tan(r_n)$

Then

$$\tan(\sum_{i=1}^{n} r_i) = -\frac{\sum_{i=0}^{\mathrm{int}((n-1)/2)} (-1)^i a_{2i+1}}{\sum_{i=0}^{\mathrm{int}(n/2)} (-1)^i a_{2i}} = \frac{\{1\} - \{3\} + \{5\} - \cdots}{1 - \{2\} + \{4\} - \cdots} = -\frac{a_1 - a_3 + a_5 - \cdots}{1 - a_2 + a_4 - \cdots}$$

# Lemma

- First note that $a_j = (-1)^j \{j\}$ That proves the last equality of the theorem.
- Lemma: Let $s_i$=tan($r_i$) for all $i$= 1, …, $n$. Then

$$\{p\}_n + \{p-1\}_n s_{n+1} = \{p\}_{n+1}$$

$$\{n\}_n s_{n+1} = \{n+1\}_{n+1}$$

- Proof. Does a term in $\{p\}_{n+1}$ contain $s_{n+1}$ or not?
  - The terms that do have $s_{n+1}$ in it have $p$-1 variables in it along with $s_{n+1}$. This gives $\{p\text{-}1\}_n$ $s_{n+1}$ .
  - The terms that do not have $s_n$ are precisely the terms in $\{p\}_n$.
  - Adding these together gives line 1 of the lemma.
- For the second line, $\{n\}_n$ is simply the product of the $s$'s up to $n$. Multiplying this by $s_{n+1}$ gives the product up to $n$+1.

# Proof of theorem

- By induction. If $n$=1, then we get $s_1$ = $s_1$, or tan($r_1$)=tan($r_1$), which is true.
- Suppose theorem true up to $n$. Then we apply the lemma termwise (included subscripts)

$$\tan(\sum_{i=1}^{n+1} r_i) = \tan(\sum_{i=1}^{n} r_i + r_{n+1})$$

$$= \frac{\dfrac{\{1\}_n - \{3\}_n + \{5\}_n \cdots}{1 - \{2\}_n + \{4\}_n \cdots} + s_{n+1}}{1 - \dfrac{\{1\}_n - \{3\}_n + \{5\}_n \cdots}{1 - \{2\}_n + \{4\}_n \cdots} s_{n+1}} = \frac{\{1\}_n - \{3\}_n + \{5\}_n \cdots + (1 - \{2\}_n + \{4\}_n \cdots) s_{n+1}}{1 - \{2\}_n + \{4\}_n \cdots - (\{1\}_n - \{3\}_n + \{5\}_n \cdots) s_{n+1}}$$

$$= \frac{(\{1\}_n + 1 s_{n+1}) - (\{3\}_n + \{2\}_n s_{n+1}) + (\{5\}_n + \{4\}_n s_{n+1}) - \cdots}{(1 - (\{2\}_n + \{1\}_n s_{n+1}) + (\{4\}_n + \{3\}_n s_{n+1}) - \cdots}$$

$$= \frac{\{1\}_{n+1} - \{3\}_{n+1} + \{5\}_{n+1} \cdots}{1 - \{2\}_{n+1} + \{4\}_{n+1} \cdots}$$

- Result is the formula for $n$+1.

# The original problem

- Evaluate $$\arctan(1) + \arctan(2) + \arctan(3)$$
- Evaluate first the tangent of this in terms of the tangents of the original terms

$$\tan(\arctan(1)) = 1; \tan(\arctan(2)) = 2; \tan(\arctan(3)) = 3$$

- Find equation that has 1, 2,3 as roots

$$(x-1)(x-2)(x-3) = 0$$

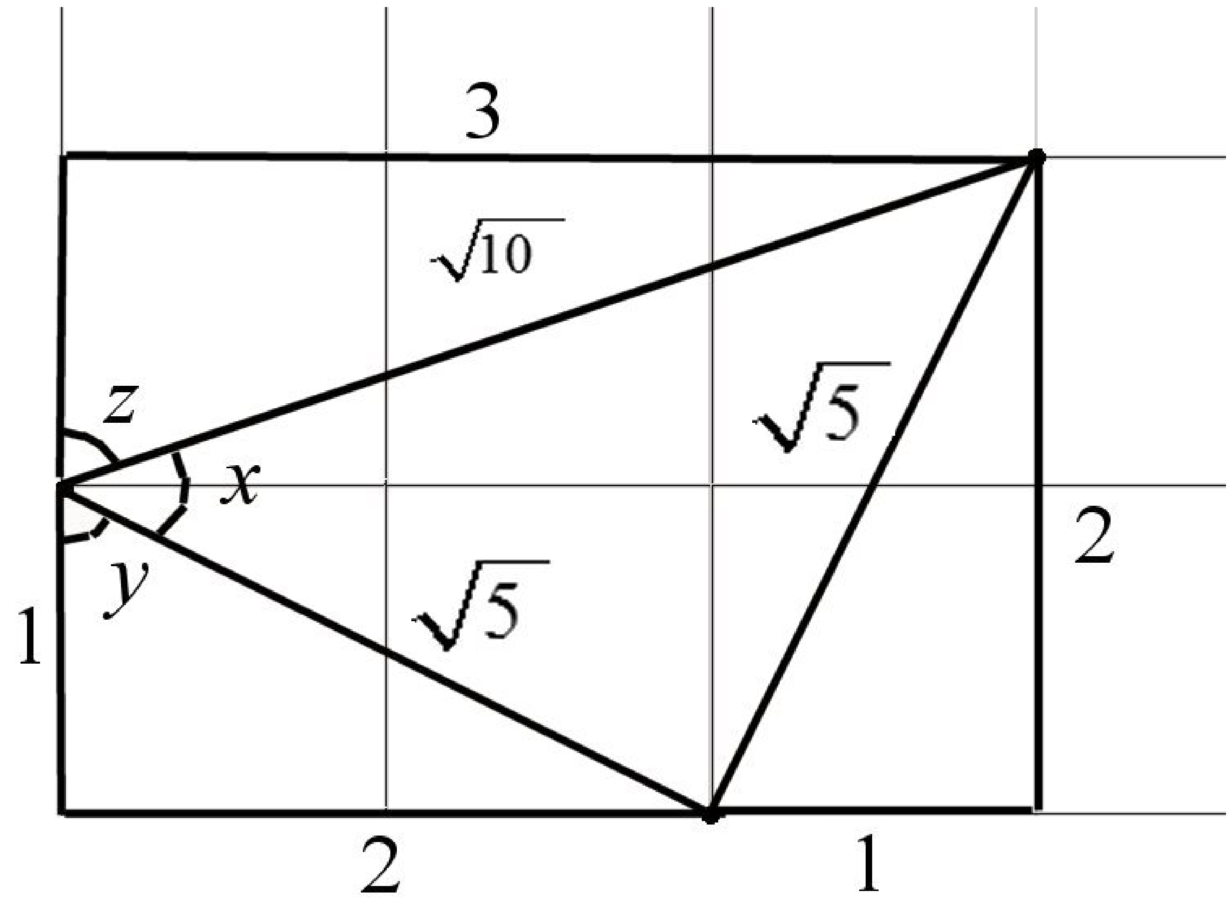
$$x^3 - 6x^2 + 11x - 6 = 0$$

- Apply formula

$$\frac{\{1\} - \{3\}}{1 - \{2\}} = -\frac{a_1 - a_3}{1 - a_2} = -\frac{(-6) - (-6)}{1 - 11} = -\frac{0}{-10} = 0$$

- tan($x$+$y$+$z$)=0, so that $x+y+z = 0 + n\pi$ for some $n$. The individual angles are all between $\pi/4$ and $\pi/2$, so their sum cannot exceed $3\pi/2$. This implies $n$=1 and $x+y+z=\pi$
- That is the answer to the original problem.

# Geometric Solution



# Example 2

$$\tan(\arctan(2\cos(2\pi/9)+\arctan(2\cos(8\pi/9)+\arctan(2\cos(14\pi/9))$$

- tan(arctan(2cos(2π/9))= 2cos(2π/9)
- Use triple angle formula

$$\cos(\pi/3)=4\cos^3(\pi/9)-3\cos(\pi/9)=-\frac{1}{2}$$

$$8\cos^3(\pi/9)-6\cos(\pi/9)+1=0; x=2\cos(\pi/9)$$

$$x^3-3x+1=0$$

$$\frac{\{1\}-\{3\}}{1-\{2\}}=-\frac{a_1-a_3}{1-a_2}=-\frac{0-1}{1-(-3)}=\frac{1}{4}$$

# Example 3

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**Proof Without Words: An Arctangent Identity**

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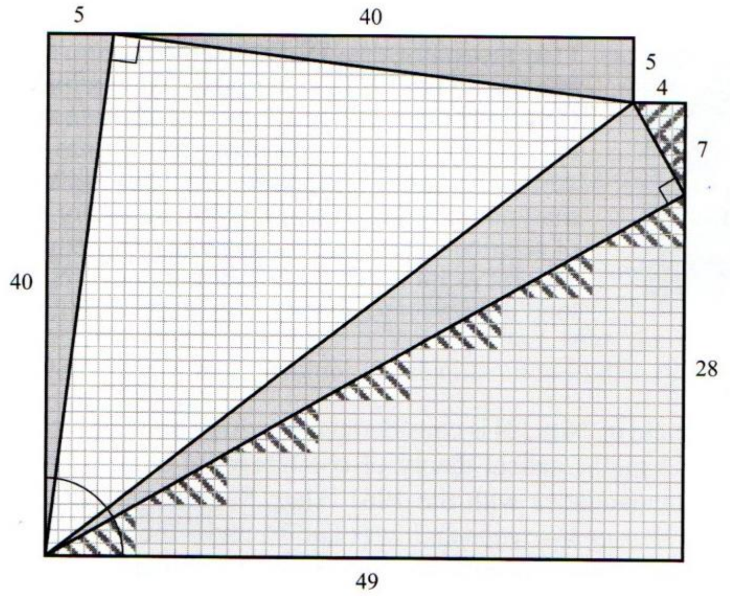



$$\arctan\frac{1}{8}+\frac{\pi}{4}+\arctan\frac{1}{7}+\arctan\frac{4}{7}=\frac{\pi}{2}.$$

Call this quantity U

# Example 3

College Mathematical Journal, 2014 November, p. 357

$$(x-\frac{1}{8})(x-1)(x-\frac{1}{7})(x-\frac{4}{7})$$

$$=x^4-\frac{103}{56}x^3+\frac{99}{98}x^2-\frac{71}{392}x+\frac{1}{98}$$

$$\tan(U)=\tan(\arctan(\frac{1}{8})+\arctan(1)+\arctan(\frac{1}{7})+\arctan(\frac{4}{7}))$$

$$=\frac{\{1\}-\{3\}}{1-\{2\}+\{4\}}=-\frac{a_1-a_3}{1-a_2+a_4}$$

$$=-\frac{-\frac{103}{56}-\frac{-71}{392}}{1-\frac{99}{98}+\frac{1}{98}}=\frac{325/196}{0}=\infty$$

$$U=\arctan(\infty)=\frac{\pi}{2}$$

# Possible avenues for research

- Newton’s Identities – how do they relate to this problem?
- Something similar for sines and cosines? Two trig functions to work with
- How does the formula relate to the geometric solution?
- Lill’s method also deals with tangents and polynomial equations. How does it relate to this problem?

# Acknowledgements


- Alfredo Kraus in Yahoo Answers obtained the same solution as in this presentation but did not relate it to polynomial equations
  - https://answers.yahoo.com/question/index?qid=20110119025857AAzOMuo

# Newton's Identities

in bracket and brace notation

$$\sum_{j=0}^{n} (-1)^{j} \{j\}[k-j] = 0 \qquad \text{for } k \geq n$$

$$\sum_{j=0}^{k} (-1)^{j} \{j\}[k-j] = (-1)^{k} (n-k)\{k\} \qquad \text{for } k < n$$

# Lill's Method

$x^3+6x^2+11x+6=0$

# Triangle Identity

- Found this on Math Stack Exchange:
  - Show tan(A)+tan(B)+tan(C) = tan(A)tan(B)tan(C)
  - if a+b+c=180 degrees; e.g. they are the angles of a
  - triangle
- Solution:

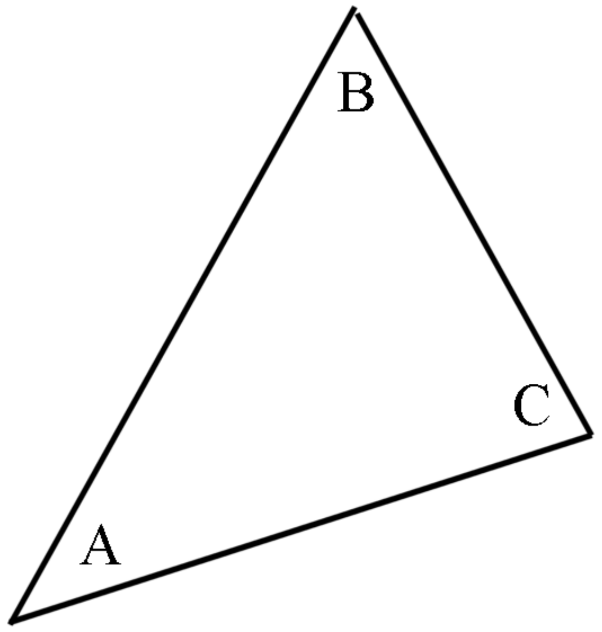
$$0 = \tan(\pi) = \tan(A+B+C) = \frac{\{1\}-\{3\}}{1-\{2\}}$$

$$\{1\}-\{3\} = 0$$

$$\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C) = 0$$

$$\tan(A)+\tan(B)+\tan(C) = \tan(A)\tan(B)\tan(C)$$

$$QED$$



# Example 4

$$\tan(\arctan(\frac{1}{2}(\sqrt{5}+\sqrt{7+2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(\sqrt{5}-\sqrt{7+2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(-\sqrt{5}+\sqrt{7-2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(-\sqrt{5-\sqrt{7-2\sqrt{5}}})))=?$$

# Example 4

$$\tan(\arctan(\frac{1}{2}(\sqrt{5}+\sqrt{7+2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(\sqrt{5}-\sqrt{7+2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(-\sqrt{5}+\sqrt{7-2\sqrt{5}}))$$
$$+\arctan(\frac{1}{2}(-\sqrt{5-\sqrt{7-2\sqrt{5}}}))) = -5/6$$

# Example 4

$$\tan(\arctan(\frac{1}{2}(\sqrt{5}+\sqrt{7+2\sqrt{5}}))+\arctan(\frac{1}{2}(\sqrt{5}-\sqrt{7+2\sqrt{5}}))+\arctan(\frac{1}{2}(-\sqrt{5}+\sqrt{7-2\sqrt{5}}))+\arctan(\frac{1}{2}(-\sqrt{5}-\sqrt{7-2\sqrt{5}})))$$

- One can compute by adding/multiplying these roots together that they solve this equation:

$$x^4-6x^2-5x-1=0$$

- Apply the formula:

$$\frac{\{1\}-\{3\}}{1-\{2\}+\{4\}}=-\frac{a_1-a_3}{1-a_2+a_4}-\frac{0-(-5)}{1-(-6)+(-1)}=-\frac{5}{6}$$