Calculating the Surface Area of Smooth Manifolds Embedded in 4 Dimensions

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Outline

- Getting Undergraduates Interested in Mathematics

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- Questions asked by curious students

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- 2-dimensional volume in 4 dimensional space
- An example

Area (2-dimensional volume) in $\mathbb{R}^3$

Area of the parallelogram P spanned by two vectors
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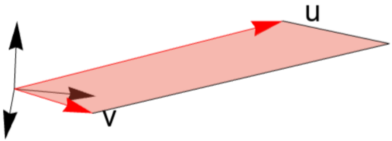

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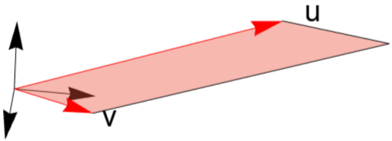

$$Vol\,(\mathbf{u}, \mathbf{v}) = \|u \times v\| \qquad \text{where} \qquad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Question Posed by the Curious Student:

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Consider $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$:

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How do I calculate the area, $Vol\,(\mathbf{u}, \mathbf{v})$, of the parallelogram spanned by $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^4$?

(And for the even clarioser, the $n - k$ dimensional volume of a parallelotope in $\mathbb{R}^n$.)

Volumes and the Gram Matrix

Definition

Given a set of k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$, the Gram Matrix $G_{\mathbf{u}_1,\ldots,\mathbf{u}_k}$ associated with these vectors is the matrix with entries given by taking the pairwise inner products of the vectors $\mathbf{u}_i$

$$(G_{\mathbf{u}_1,\ldots,\mathbf{u}_k})_{i,j} = \mathbf{u}_i \cdot \mathbf{u}_j.$$

Now how can we use this?

Volumes and the Gram Matrix

One Possible way: We can still calculate dot products of vectors in $\mathbb{R}^4$, so the *Gram Matrix* is well defined:

$$G_{u,v} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{bmatrix}$$

In fact this will be well defined when $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ for any $n \in \mathbb{N}$.

A useful result

Theorem

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$*, with the standard inner product. Then*

$$Vol\left(\mathbf{u}_1, \ldots, \mathbf{u}_k\right) = \left(\det\left(G_{\mathbf{u}_1, \ldots, \mathbf{u}_k}\right)\right)^{1/2}$$

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Corrollary

Let $\mathbf{u}$,$\mathbf{v}$ *be two vectors in* $\mathbb{R}^n$*, and* $G_{\mathbf{u},\mathbf{v}}$ *be the Gram matrix determined by* $\mathbf{u}$ *and* $\mathbf{v}$*. Then*

$$Vol\left(\mathbf{u}, \mathbf{v}\right) = \sqrt{\det\left(G_{\mathbf{u},\mathbf{v}}\right)}.$$

A useful result

Proof.

- Let $V = \text{span}\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^4$
- Set $\{\mathbf{e}_1, \mathbf{e}_2\}$ to be an orthonormal basis of V.

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Proof.

- Let $V = \text{span}\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^4$
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- $G_{\mathbf{u},\mathbf{v}} = P^t P \implies \det(G_{\mathbf{u},\mathbf{v}}) = \det(P^t) \det(P)$.

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- $G_{\mathbf{u},\mathbf{v}} = P^t P \implies \det(G_{\mathbf{u},\mathbf{v}}) = \det(P^t)\det(P)$.
- Now take square roots.

$\square$

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which determine a surface (a torus) embedded in $\mathbb{R}^4$. A convenient parametrization is given by

$$(x, y, z, w) = r\,(\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

$$0 \le \theta, \phi \le 2\pi$$

An elementary example

Now, how to integrate a function over this surface?
Formally we take the limit of a riemann sum

$$\lim_{\Delta\theta,\Delta\phi\to 0} \sum_{(\theta_i,\phi_i)} f(\theta_i,\phi_i)\, A\left(\Delta\theta \cdot r_\theta(\theta_i,\phi_i), \Delta\phi \cdot r_\phi(\theta_i,\phi_i)\right)$$

where $A(\vec{u},\vec{v})$ is the area of the parallelogram determined by the vectors $\vec{u},\vec{v} \in \mathbb{R}^4$.

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where $A(\vec{u},\vec{v})$ is the area of the parallelogram determined by the vectors $\vec{u},\vec{v} \in \mathbb{R}^4$.
In the limit this equals the surface integral

$$\int_0^{2\pi}\int_0^{2\pi} f(\theta,\phi)\, A(r_\theta, r_\phi)\, d\theta d\phi$$

An elementary example

Now we can use the Gram matrix to find the area element:

$$r_\theta = \begin{pmatrix} -\sin\theta & \cos\theta & 0 & 0 \end{pmatrix}$$
$$r_\phi = \begin{pmatrix} 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

and

$$G_{r_\theta, r_\phi} = \begin{vmatrix} r_\theta \cdot r_\theta & r_\theta \cdot r_\phi \\ r_\phi \cdot r_\theta & r_\phi \cdot r_\phi \end{vmatrix}$$

which at the end of the day equals just the determinant of the identity matrix, or just 1. In this case the surface area is

$$\int_0^{2\pi} \int_0^{2\pi} 1 \, d\theta d\phi = 4\pi^2.$$

A more complicated (yet illustrative) example

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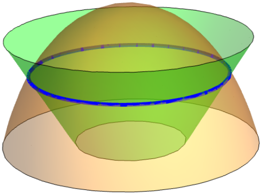

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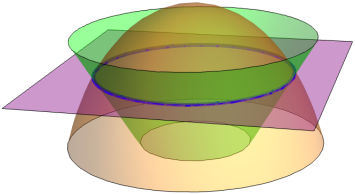

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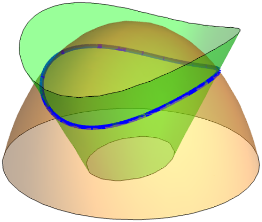
But this isn't very interesting since C lies entirely in a 2-dimensional plane within $\mathbb{R}^3$.

A deformation of the circle

Here we have slightly deformed C so that it does not lie in a 2-dimensional hyperplane in $\mathbb{R}^3$:



$$\begin{aligned} 1 &= x^2 + y^2 + z^2 \\ z^2 &= a\,x^2 + y^2 \end{aligned}$$

$$a \neq 1$$

In this picture $a = 2$.

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An embedding of the 2-sphere in $\mathbb{R}^4$ with a deformation parameter a:

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and after some work a parametrization $r(\theta, \phi) = (x(\theta, \phi), \dots)$ with

$$\begin{aligned} x &= \frac{1}{\sqrt{a+1}} \cos\phi \\ y &= \frac{1}{\sqrt{2}} \cos\theta \sin\phi \\ z &= \frac{1}{\sqrt{2}} \sin\theta \sin\phi \\ w &= \left(\frac{a}{a+1} \cos^2\phi + \frac{1}{2} \sin^2\phi \right)^{1/2} \end{aligned}$$

Calculation of surface differential element (by Gram matrix)

The tangents to the coordinate curves:

$$r_\theta = \left(0, -\frac{1}{\sqrt{2}}\sin\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\theta\sin\phi, 0\right)$$

$$r_\phi = \Bigg(-\frac{1}{\sqrt{a+1}}\sin\phi, \quad \frac{1}{\sqrt{2}}\cos\theta\cos\phi,$$

$$\frac{1}{\sqrt{2}}\sin\theta\cos\phi, \frac{\left(\frac{1}{2}-\frac{a}{a+1}\right)\sin\phi\cos\phi}{\left(\frac{a}{a+1}\cos^2\phi + \frac{1}{2}\sin^2\phi\right)^{1/2}}\Bigg)$$

Calculation of surface differential element (by Gram matrix)

$$\sqrt{\det\left(G_{r_\theta,r_\phi}\right)} = \begin{vmatrix} r_\theta \cdot r_\theta & r_\theta \cdot r_\phi \\ r_\phi \cdot r_\theta & r_\phi \cdot r_\phi \end{vmatrix}^{1/2}$$

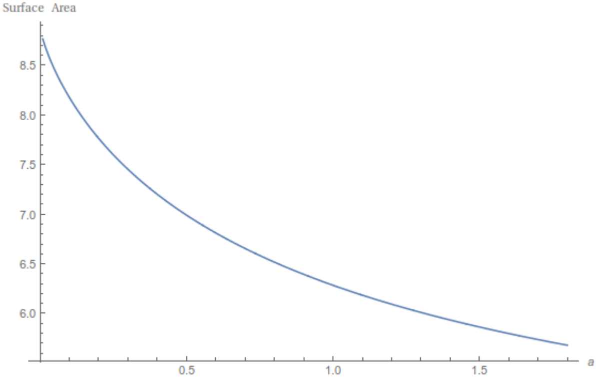
$$= \begin{vmatrix} \sin^2\phi & \frac{1}{2}\sin\theta\cos\theta\sin\phi\cos\phi \\ \frac{1}{2}\sin\theta\cos\theta\sin\phi\cos\phi & \frac{(a+1)+(a-1)\cos(2\phi)}{(3a+1)+(a-1)\cos(2\phi)} \end{vmatrix}^{1/2}$$

$$= (\text{Something very long})^{1/2}$$

Some numerical evaluations for selected values of a

$$S = \int_0^{\pi} \int_0^{2\pi} \sqrt{\det\left(G_{r_\theta, r_\phi}\right)}\, d\theta d\phi$$

a	Surface Area
0.2	7.76546
0.4	7.20102
0.6	6.81061
0.8	6.51623
1.0	6.28319
1.2	6.09261



Thank You!