Non-Special Divisors on Graphs

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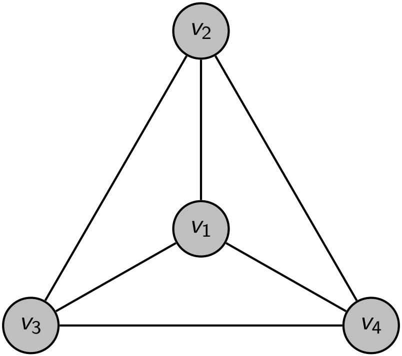
Consider a connected graph G which has no loops, but which may have multi-edges.

The **genus** of G is

$$g = \text{ number of edges} - \text{ number of vertices} + 1.$$

K_4

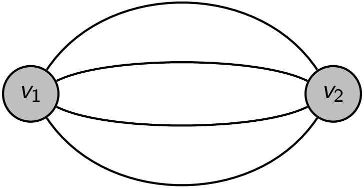
Example:



$$g = 6 - 4 + 1 = 3$$

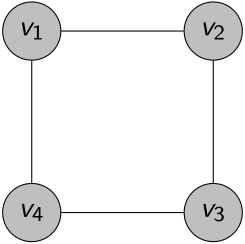
Dipole Graph B_4

Example:



$$g = 4 - 2 + 1 = 3$$

Example:



$$g = 4 - 4 + 1 = 1$$

A **divisor** D on G is an assignment of an integer a_v to each vertex v,

$$D = \sum a_v v.$$

Interpretation:

If $a_v > 0$, then a_v represents a number of chips at vertex v (assets).

If $a_v < 0$, then vertex v is in debt by a_v chips.

If no vertex of D is in debt, we say D is **effective** (written $D \geq 0$).

The **degree** of D is $\sum a_v$ (the net worth).

Example: $G = K_4$

$$D_1 = -v_1 + v_3 + 3v_4$$

$$D_2 = v_2 + 2v_3$$

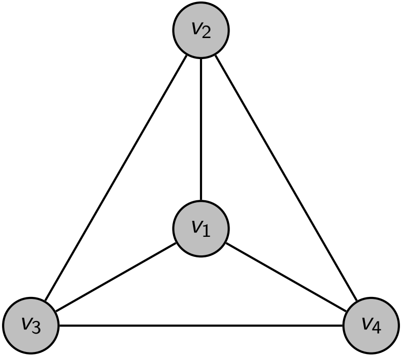
The divisor D_2 is effective but the divisor D_1 is not.

Both divisors have degree 3.

Chip-firing move on a vertex of D

We fire a vertex v by sending a chip along each edge e adjacent to v to the other endpoint of e.

Example: $G = K_4$, $D_1 = -v_1 + v_3 + 3v_4$



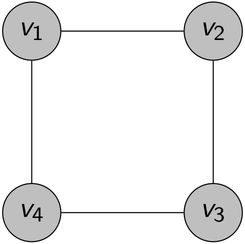
Firing v_4 sends a chip from v_4 to each of v_1, v_2, and v_3.

D_1 changes to $D_2 = v_2 + 2v_3$.

Set-firing move on D

We fire a set W of vertices by sending a chip along each edge from W to its complement.

Example (C_4): Let $D_1 = -2v_1 + v_2 + v_3$ and let $W = \{v_2, v_3\}$.



Firing the set W transforms D_1 into $D_2 = -v_1 + v_4$.

Note that firing the complement $W^c = \{v_1, v_4\}$ on D_2 transforms D_2 back to D_1.

If we fire all vertices at once, there is no change in the divisor.

Linear Equivalence of Divisors

If D_1 and D_2 can be transformed into one another by chip-firing moves, we say that D_1 and D_2 are **linearly equivalent** (written $D_1 \sim D_2$).

We look at equivalence classes of divisors under linear equivalence.

Example: There are 4 linear equivalence classes of degree 0 divisors on C_4.

Representatives are

$$0, \qquad -v_1 + v_2, \qquad -v_1 + v_3, \qquad -v_1 + v_4$$

The divisor 0 is an effective divisor of degree 0.

The other three equivalence classes contain no effective divisors.

Non-Special Divisors

A divisor of degree $g - 1$ is called **non-special** if it is not linearly equivalent to any effective divisor.

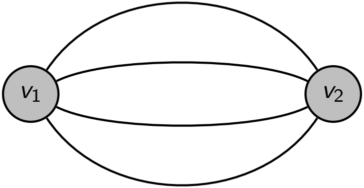
Example: The following three divisors on C_4 are non-special:

$$-v_1 + v_2$$

$$-v_1 + v_3$$

$$-v_1 + v_4$$

Non-Special Divisors on B_4



Any chip-firing move must send $4k$ chips from one vertex to the other, for some integer k.

Consider divisors of degree $g - 1 = 2$ on B_4. There are four linear equivalence classes, with representatives

$$2v_1, \qquad v_1 + v_2, \qquad 2v_2, \qquad -v_1 + 3v_2$$

The non-special divisors are those linearly equivalent to $-v_1 + 3v_2$.

Non-Special Divisors and the Tutte Polynomial

The following result may be proved using Dhar's burning algorithm - see Cori - Le Borgne, Merino.

Let $T(x, y)$ be the Tutte polynomial of G.

Then $T(1, 0)$ is the number of linear equivalence classes of non-special divisors on G.

Remark: It is well-known that $T(1, 1)$ is the number of spanning trees of G.

Example: The Tutte polynomial of C_4 is

$$T(x, y) = x^3 + x^2 + x + y.$$

$$T(1, 0) = 3$$

Example: The Tutte polynomial of B_4 is

$$T(x, y) = x + y + y^2 + y^3.$$

$$T(1, 0) = 1$$

Example: The Tutte polynomial of K_4 is

$$T(x, y) = x^3 + 3x^2 + 2x + 4xy + y^3 + 3y^2 + 2y.$$

$$T(1, 0) = 6$$

Baker - Norine Construction of Non-Special Divisors ν_P

Let $P = (v_1, v_2, \cdots, v_n)$ be an ordering of the vertices of G.

Let A_{ij} be the number of edges between v_i and v_j.

Set

$$\nu_P = -v_1 + (A_{21} - 1)v_2 + (A_{31} + A_{32} - 1)v_3 + \cdots + (A_{n1} + A_{n2} + \cdots + A_{n,n-1} - 1)v_n.$$

The divisor ν_P is non-special.

Every non-special divisor is equivalent to a divisor of the form ν_P, for some P.

Proof that ν_P is non-special

Suppose that ν_P is not non-special, i.e., that ν_P is linearly equivalent to an effective divisor D.

Then we can transform D into ν_P by some firing amounts $x_1, x_2, \cdots, x_n$, where x_i is the number of times v_i is fired. We can assume that $x_i \geq 0$ for all i and $x_i = 0$ for at least one i, since firing all vertices at once does not change D.

Let v_k be the lowest index vertex in D that is not fired. The vertex v_k starts with a nonnegative quantity of chips and gains chips but does not lose chips. In particular, v_k gains at least

$$A_{k1} + A_{k2} + \cdots + A_{k,k-1}$$

chips, since $v_1, v_2, \cdots, v_{k-1}$ are all fired at least once, by assumption. This is a contradiction, because ν_P has only

$$A_{k1} + A_{k2} + \cdots + A_{k,k-1} - 1$$

chips at v_k.

Non-Special Divisors on K_4

Choose an ordering P on the vertices of K_4.

For the graph K_4,

$$A_{ij} = 1 \text{ for all } i \neq j.$$

Then

$$\nu_P = -v_1 + 0v_2 + v_3 + 2v_4$$

Note that we can obtain all 6 linear equivalence classes of non-special divisors on K_4 by fixing v_1 and permuting the other three vertices.

Note also that v_1 is the only vertex in debt, and that no non-empty subset of the other vertices can be fired without another vertex going into debt. The divisor shown is an example of a v_1-**reduced** divisor.

The Canonical Divisor K on a graph G

Let $\deg(v)$ be the **degree** of the vertex v, i.e., the number of edges incident to v.

The **canonical divisor** on G is

$$K = \sum_{v} (\deg(v) - 2)v$$

The degree of K is $2g - 2$.

Example: The canonical divisor on K_4 is $K = v_1 + v_2 + v_3 + v_4$.

Proposition: (Baker-Norine) If D has degree $g-1$, then D is non-special if and only if $K-D$ is non-special.

Example: On K_4, the divisor $D = -v_1 + v_3 + 2v_4$ is non-special.

The canonical divisor is $K = v_1 + v_2 + v_3 + v_4$.

Then

$$K - D = 2v_1 + v_2 - v_4$$

which is also non-special.

Riemann-Roch Dimension of a Divisor D

The **dimension** $r(D)$ of a divisor D is defined as follows.

If D is not linearly equivalent to any effective divisor, $r(D) = -1$.

Otherwise, if D is linearly equivalent to some effective divisor, $r(D)$ is the largest nonnegative integer d such that $D - E$ is linearly equivalent to some effective divisor, for all effective divisors E of degree d.

Baker-Norine Characterization of $r(D)$

Let

$$\deg^+(D) = \sum_{a_v > 0} a_v$$

(the degree of the effective part of D).

Then

$$r(D) = -1 + \min\{\deg^+(D' - \nu) \mid D' \sim D \text{ and } \nu \text{ is non-special}\}.$$

Example: B_4

Let $D = v_1 + 2v_2$. Recall that $\nu = -v_1 + 3v_2$ is non-special.

$$D - \nu = 2v_1 - v_2$$

Every divisor linearly equivalent to $D - \nu$ is of the form

$$(2 + 4k)v_1 + (-1 - 4k)v_2$$

The minimum value of $\deg^+((2 + 4k)v_1 + (-1 - 4k)v_2)$ is 2.

Therefore $r(D) = 1$.

The Riemann-Roch Theorem for Graphs

(Baker - Norine, 2007)

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

Non-special divisors are central to Baker and Norine's proof.

Example: B_4

Example: The canonical divisor on B_4 is $K = 2v_1 + 2v_2$.

Let $D = v_1 + 2v_2$. Recall that $r(D) = 1$. Recall also that $\nu = -v_1 + 3v_2$ is non-special.

Then $K - D = v_1$ and $K - D - \nu = 2v_1 - 3v_2$. Every divisor linearly equivalent to $K - D - \nu$ is of the form

$$(2 + 4k)v_1 + (-3 - 4k)v_2$$

It follows that $r(K - D) = 0$.

$$r(D) - r(K - D) = 1 - 0 = 1$$

$$\deg(D) + 1 - g = 3 + 1 - 3 = 1$$

Other Interpretations of Non-Special Divisors

Non-special divisor classes correspond to maximal G-parking functions (see Benson (2008), Biggs (1999), Greene and Zaslavsky (1983)).

Non-special divisor classes correspond to acyclic orientations on G with a unique source.

Using Dhar's burning algorithm with a fixed ordering on edges, non-special divisor classes correspond to spanning trees of G with no externally active edges.