Monthly Problem 3173, Samuel Beatty, and $\frac{1}{p} + \frac{1}{q} = 1$

Ezra Brown
Virginia Tech

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What to expect

- An interesting pair of sequences
- Problem 3173 of the *American Mathematical Monthly*
- Beatty sequences
- Samuel Beatty
- The Proof
- Beatty sequences could be called **Wythoff** sequences ...
- ... or even **Rayleigh** sequences.
- Another appearance of $\frac{1}{p} + \frac{1}{q} = 1 \ldots$ coincidence?

An interesting pair of sequences

Let $p = (1+\sqrt{5})/2 = \phi$ and $q = \phi/(\phi - 1)$.

Let $A = \{\lfloor np \rfloor : n = 1, 2, 3, \ldots\}$ and $B\{\lfloor nq \rfloor : n = 1, 2, 3, \ldots\}$. Then

$$A = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots\}, \text{ and}$$

$$B = \{2, 5, 7, 10, 13, 15, 18, 20, \ldots\}$$

Looks like the two sequences will contain all the positive integers without repetition.

It also happens that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Coincidence?

3173. Proposed by Samuel Beatty, University of Toronto.

If x is a positive irrational number and y is its reciprocal, prove that the sequences

$$(1+x), 2(1+x), 3(1+x), \dots \text{ and}$$
$$(1+y), 2(1+y), 3(1+y), \dots$$

contain one and only one number between each pair of consecutive positive integers.

Equivalent statement: If x is a positive irrational number and y is its reciprocal, prove that the sequences

$$\lfloor (1+x) \rfloor, \lfloor 2(1+x) \rfloor, \lfloor 3(1+x) \rfloor, \dots \text{ and}$$
$$\lfloor (1+y) \rfloor, \lfloor 2(1+y) \rfloor, \lfloor 3(1+y) \rfloor, \dots$$

contain each positive integer once without duplication.

Problem 3173 rephrased

Observation: if $p = 1 + x$ and $q = 1 + y = 1 + 1/x$, then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{1+x} + \frac{x}{x+1} = \frac{1+x}{1+x} = 1.$$

The common rephrasing of Beatty's problem is as follows:

If p and q are irrational numbers greater than 1 such that $1/p + 1/q = 1$, then the sequences

$$A = \{\lfloor p \rfloor, \lfloor 2p \rfloor, \lfloor 3p \rfloor, \ldots\} \text{ and } B = \{\lfloor q \rfloor, \lfloor 2q \rfloor, \lfloor 3q \rfloor, \ldots\}$$

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This is far from obvious.

Beatty sequences

A *Beatty sequence* is a sequence of integers of the form $\{\lfloor np \rfloor : n = 1, 2, \ldots\}$, where p is a positive irrational number.

Example: $\alpha = \sqrt{2}$ generates the Beatty sequence

$$A = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, \ldots\}.$$

The *complement* of the Beatty sequence A is the sequence

$$B = \mathbb{Z}^{+} - A = \{3, 6, 10, 13, 17, 20, 23, 27, 30, 34, 37, 40, 44, \ldots\}.$$

The big surprise is that this complement $B = \{\lfloor n\sqrt{2}/(\sqrt{2} - 1) \rfloor : n = 1, 2, \ldots\}$ is also a Beatty sequence.

The Main Theorem

Theorem: If p is irrational and $p > 1$, then the complement of the Beatty sequence generated by p is the Beatty sequence generated by q, where $1/p + 1/q = 1$.

Problem 3173 asks us to prove this theorem.

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Let's meet Samuel Beatty first.

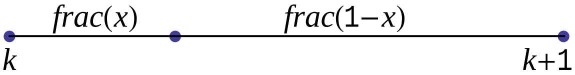
Who Was Samuel Beatty?

- b. 1881, Owen Sound, Ont.; entered U of Toronto 1903 as student, never left.
- 1915 Math PhD – the only doctoral student of John Charles Fields, he of the Fields Medal. Joined the math faculty at Toronto, eventually Head, Dean of the Faculty, and University Chancellor.
- Instrumental in bringing Donald Coxeter and Richard Brauer to Toronto as faculty members in the 1930s.
- A beloved teacher, mentor, and strong supporter of his students.
- Appropriately, he is most remembered for the problem he published in the flagship journal of some obscure organization whose mission is **"to advance the mathematical sciences, especially at the collegiate level."**
- Hooray for Sam Beatty!

Solution of 3173 by A. Ostrowski (Göttingen) and J. Hislop (Glasgow).

Lemma 1

- Let $frac(x) = x - \lfloor x \rfloor$ be the fractional part of x. If x is not an integer, then $frac(x) + frac(1 - x) = 1$.
- Here's a picture:



Lemma 2

If j is an integer, then $frac(x) = frac(j + x)$ for every integer j.

The Proof

Let p and q be irrational with $1/p + 1/q = 1$. Let's count the number of elements in A and B that do not exceed some positive integer n.

- For k and n integers, $kp < n$ if and only if $k \leq \lfloor n/p \rfloor < n/p$.
- Thus there are $\lfloor n/p \rfloor + \lfloor n/q \rfloor$ elements of A and B less than n. This might include duplications.
- $$\begin{aligned} n = n/p + n/q &= \lfloor n/p \rfloor + \lfloor n/q \rfloor + frac(n/p) + frac(n - n/p) \\ &= \lfloor n/p \rfloor + \lfloor n/q \rfloor + frac(n/p) + frac(1 - n/p), \text{ by Lemma 1} \\ &= \lfloor n/p \rfloor + \lfloor n/q \rfloor + 1, \qquad \text{by Lemma 2.} \end{aligned}$$

The Proof, continued

- Thus, there are $\lfloor n/p \rfloor + \lfloor n/q \rfloor = n - 1$ elements of A and B less than n.

- Similarly, there are n elements of A and B less than $n + 1$.

- Thus, $n - (n - 1) = 1$ element in A and B is in $[n, n + 1)$, and there are no duplications – and we're done.

- But two decades earlier, we find . . .

. . . Wythoff's Game (1907)

- Two players, two stacks of chips.
- Players alternately take either any number of chips from one stack or equal numbers from both stacks.
- Objective: to take the last chip.
- Find a winning strategy.

Wythoff's Game: a strategy

Let's look at the complementary Beatty sequences A and B, generated by $p = \phi$ and $q = \phi/(\phi - 1)$:

$$\begin{array}{llllllllllllll} A: & 1 & 3 & 4 & 6 & 8 & 9 & 11 & 12 & 14 & 16 & 17 & 19 & 21 & \dots \\ B: & 2 & 5 & 7 & 10 & 13 & 15 & 18 & 20 & 23 & 26 & 28 & 31 & 34 & \dots \end{array}$$

There is a connection between the Beatty sequences A and B and Wythoff's Game.

Wythoff's Game: a strategy

The connection between the Beatty sequences A and B and Wythoff's Game is this:

If you move so that the numbers in the two stacks are corresponding members of the sequences A and B, you will always win.

That is, the losing positions in Wythoff's Game are precisely the positions with m chips in one stack and n chips in the other, where

$$(m, n) \text{ or } (n, m) = (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), \ldots$$

We could call Beatty sequences *Wythoff sequences* ...

Lord Rayleigh's theorem (1894)

... or even Rayleigh sequences.

Rayleigh's Theorem states that when a constraint is introduced to a vibrating system, the new frequencies of vibration interleave the old frequencies. Here is his example:

> *"If x be an incommensurable number less than unity, one of the series of quantities m/x, $m/(1-x)$, where m is a whole number, can be found which shall lie between any given consecutive integers, and but one such quantity can be found." (From The Theory of Sound, 2nd ed., vol 1, pp.122-123)*

Look familiar?

Many Connections

- Number Theory
- Combinatorial Games
- Electron Diffraction
- Quasicrystals
- Penrose Tilings
- Digital Signal Processing

... and finally ...

Young's Inequality (1912) and its deep consequences

Let $p > 1$ and let $1/p + 1/q = 1$. Then for all nonnegative real numbers a and b,

$$ab \leq \frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}.$$

This leads to *Minkowski's inequality, Hölder's inequality,* and the foundational theorem about the vector spaces $\mathcal{L}_p$ of p'th-power *Lebesgue-integrable functions* and their *dual* spaces of real-valued continuous linear mappings, namely:

The dual of $\mathcal{L}_p$ is $\mathcal{L}_q$, where $1/p + 1/q = 1$.

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Who'dathunkit?

THANK YOU!