Five families and a well: A new look at an old problem

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What To Expect

- The *Nine Chapters on the Mathematical Art*: anticipating modern methods of solving linear systems
- 5 Families, Chapter 8, Problem 13: five equations in six unknowns
- The smallest integer values for the longest well-rope and the well depth
- An “Aha!” moment
- Derangements – and a conjecture
- Matt Crawford – and a theorem
- Concluding thoughts

The *Jiuzhang suanshu*, or *Nine Chapters on the Mathematical Art*

- Where and when: Ancient China, ca. 1st century CE.
- Contents: a practical handbook of mathematics consisting of 246 problems intended to provide methods to be used to solve everyday problems of engineering, surveying, trade, and taxation.
- Historical importance: standard methods and concepts that predate Western mathematics by many centuries, such as square roots, cube roots, arithmetic with negative numbers, and solving systems of linear equations using matrices and Gaussian elimination.

Chapter 8, Problem 13: The Well Problem

- "Five families share a water well. Two of A's well-ropes are short of the water by one of B's ropes. Three of B's ropes are short of the water by one of C's ropes. Four of C's ropes are short of the water by one of D's ropes. Five of D's ropes are short of the water by one of E's ropes. Six of E's ropes are short of the water by one of A's ropes. Find the depth of the well and the length of each family's well-rope."
- We obtain the following system of 5 linear equations in 6 unknowns:

$$2A + B = w$$
$$3B + C = w$$
$$4C + D = w$$
$$5D + E = w$$
$$6E + A = w$$

- The solution method in the *Nine Chapters* anticipates Gaussian elimination by nearly two millennia.

The smallest integer solution

- The smallest positive integers that give a solution to the problem are as follows:
- $A = 265$, $B = 191$, $C = 148$, $D = 129$, $E = 76$, and $w = 721$.

What ran through my mind:

“Wait a minute . . . what’s 265 doing in the middle of a problem from Ancient China?”

Generalizing the well problem

n = number of families
$A(n)$ = length of the longest rope
$w(n)$ = depth of the well

n	$A(n)$	$w(n)$	$(n+1)!/A(n)$
1	1	3	2.00000 . . .
2	2	5	3.00000 . . .
3	9	25	2.66666 . . .
4	44	119	2.72727 . . .
5	**265**	**721**	2.71698 . . .
6	1854	5039	2.71845 . . .
7	14833	40321	2.71826 . . .
8	133496	362881	2.71828 . . .

Spot any patterns?

Derangements and the generalized Well Problem

- A *permutation* of $\{1, \ldots, n\}$ (or an n-permutation) is an ordered arrangement of $\{1, \ldots, n\}$. There are $n!$ such permutations.
- A *derangement* of $\{1, \ldots, n\}$ (or an n-derangement))is an n-permutation with no fixed elements.
- The permutation 24315 is not a derangement: neither 3 nor 5 have been moved.
- The permutation 41523 is a derangement: all five numbers have been moved.

Derangements and the generalized Well Problem

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- The derangements on four elements: $2143, 2413, 2341,$ $3142, 3412, 3421,$ $4123, 4312,$ and 4321. Thus, $D_4 = 9$.
- It happens that $D_5 = 44$ and $D_6 = 265$ (that's right). So, we have a conjecture.

The Generalized Well Conjecture

Conjecture: Let n be a positive integer. Then, for n families around a well, the length of the longest rope is equal to D_{n+1}.

Facts about derangements

Let D_n be the number of n-derangements. Then $D_1 = 0, D_2 = 1$, and for $n \geq 2$, the following statements hold:

$$D_{n+1} = (n+1)D_n + (-1)^{n+1}$$
$$D_{n+1} = n(D_n + D_{n-1})$$
$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k}$$

Enter Matt Crawford.

Matt's proof

Let F_n be the least integer length of the longest rope for n families around a well. Then

$$F_n = \left(\cfrac{1 - \cfrac{1 - \cfrac{\cfrac{1 - \cfrac{1 - \cfrac{1 - \cfrac{n!}{(n+1)!-1}\displaystyle\sum_{k=1}^{n}\cfrac{(-1)^{k-1}}{k!}}{n}}{n-1}}{\ddots}}{4}}{3}}{2} \right) ((n+1)!-1)$$

Matt's proof

$$= \frac{(n+1)!-1-\frac{(n-1)!-1-\frac{(n+1)!-1-\frac{(n+1)!-1-\frac{\left(((n+1)!-1)\left(1-\frac{n!}{(n+1)!-1}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\right)\right)}{n}}{n-1}}{\ddots}}{3}}{2}$$

which reduces to

$$F_n = (n+1)!\sum_{k=0}^{n+1}\frac{(-1)^k}{k!}.$$

In short, $F_n = D_{n+1}$, and the conjecture is a theorem!

Why the excitement?

- The problem of counting derangements first dates from Pierre Raymond de Montmort's *Essay d'analyse sur les jeux de hazard*, published in 1708.
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The bottom line

Is this a new observation? I don't know. But it got me excited and it got Matt involved in research. And Matt proved a theorem.

Maybe that's enough.

THANK YOU!