Divisibility Tests Unified

Stacking the Trimmings for Sums

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Divisibility Tests for q

Last Digit ($q = 2$, 5 and 10): $2184 \longmapsto 4$

Sum of Digits ($q = 3$ and 9): $2184 \longmapsto 2 + 1 + 8 + 4 = 15$

Alternating Sum ($q = 11$): $2184 \longmapsto 2 - 1 + 8 - 4 = 5$

"Trimming" ($q = 7$): $2184 \longmapsto 218 - 2 \cdot 4 = 210$

Talmud, Fibonacci, Pascal, Lagrange, Sylvester ...
See Dickson's *History of the Theory of Numbers*

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$$f_q : \mathbb{Z} \to \mathbb{Z}, \quad a \mapsto f_q(a)$$

such that $q \mid a \iff q \mid f_q(a)$.

$f_q(a)$ should be **easy** to compute; **easier** to check $q \mid f_q(a)$.

Trimming Test for 7

7 divides 2184 if and only if 7 divides ....

$$2184 - 21 \cdot 4 = (10 \cdot 218 + \underline{4}) - (20 \cdot 4 + \underline{1 \cdot 4})$$

$$= 10 \cdot (218 - 2 \cdot 4) \longrightarrow 218 - 2 \cdot 4$$

Notation $a = 10\bar{a} + a_0 \quad = 10(\#\text{ Dimes}) \; + \; (\#\text{ Pennies})$

$T_7(a) = \bar{a} - 2 \cdot a_0 \quad = (\#\text{ Dimes}) \; -2 \cdot (\#\text{ Pennies})$

Trimming Test for $q = 10\bar{q} + 7$

q divides a if and only if q divides

$$\begin{aligned} a - 3q \cdot a_0 &= (10\bar{a} + a_0) - 3(10\bar{q} + 7)a_0 \\ &= 10(\bar{a} - 3\bar{q}a_0) - (3 \cdot 7 - 1)a_0 \\ &= 10(\bar{a} - 3\bar{q}a_0) - 20a_0 \longrightarrow \bar{a} - (3\bar{q} + 2)a_0 =: T_q(a) \end{aligned}$$

- $T_{17}(a) = \bar{a} - 5 \cdot a_0$ e.g. $T_{17}(2184) = 218 - 5 \cdot 4$
- $T_{27}(a) = \bar{a} - (3 \cdot 2 + 2) \cdot a_0 = \bar{a} - 8 \cdot a_0$

Trimming Test for $q = 10\bar{q} + 9$

q divides a if and only if q divides ....

$$a + q \cdot a_0 = (10\bar{a} + a_0) + (10\bar{q} + 9)a_0$$

$$= 10(\bar{a} + \bar{q}a_0) + (1 + 9)a_0$$

$$= 10(\bar{a} + (\bar{q} + 1)a_0) \longrightarrow \bar{a} + (\bar{q} + 1)a_0 =: T_q(a)$$

- $T_{39}(a) == \bar{a} + 4 \cdot a_0$
- $T_{19}(a) = \bar{a} + 2 \cdot a_0$
- $T_9(a) = \bar{a} + a_0 \quad \stackrel{????}{\longrightarrow}$ **Sum of Digits**

Trimming Test for $q = 10\bar{q} + q_0$

$q =$	$\bar{q}+1$	$\bar{q}+3$	$\bar{q}+7$	$\bar{q}+9$
$T_q(a)$	$\bar{a} - \bar{q}a_0$	$\bar{a} + (3\bar{q}+1)a_0$	$\bar{a} - (3\bar{q}+2)a_0$	$\bar{a} + (\bar{q}+1)a_0$

- $T_{13}(a) = \bar{a} + 4 \cdot a_0$; $T_{23}(a) = \bar{a} + 7 \cdot a_0$
- $T_{91}(a) = \bar{a} - 9 \cdot a_0$; $T_{11}(a) = \bar{a} - a_0$

Good $T_q(a) = \bar{a} + \omega_q a_0$ w/ $\omega_q \approx \begin{cases} \frac{1}{10}q & : \quad q_0 = 1 \text{ or } 9 \\ \frac{1}{3}q & : \quad q_0 = 3 \text{ or } 7 \end{cases}$

Zeipel (1861), **Schlegel** (1876), **Zazkis** (1999) – Thanks Bud!

Trimming for 9 $\longrightarrow$ Sum of Digits for 9

Sum of Digits: $2184 \longrightarrow 2+1+8+4 = 15$

$T_9^3 : 2184 \xrightarrow{T_9} 218+4 = 222 \xrightarrow{T_9} 22+2 = 24 \xrightarrow{T_9} 2+4 = 6$

Trim then **Stack 1's**, Trim again then **Stack 1's** again, etc

$$218\color{red}{4} \xrightarrow{T_9} 218 + \color{red}{4} \overset{\textbf{Stack}}{=\!=\!=} 10 \cdot 21 + \color{red}{(8+4)}$$

$$\xrightarrow{T_9} 21 + \color{red}{(8+4)} \overset{\textbf{Stack}}{=\!=\!=} 10 \cdot 2 + \color{red}{(1+8+4)}$$

$$\xrightarrow{T_9} 2 + \color{red}{(1+8+4)} \overset{\textbf{Stack}}{=\!=\!=} \color{red}{(2+1+8+4)}$$

Trimming for $q \longrightarrow$ Weighted Sum of Digits

$$T_q(a) = \bar{a} + \omega_q a_0 \longrightarrow \underbrace{a_n + \omega_q a_{n-1} + + \cdots + \omega_q^{n-1} a_1 + \omega_q^n a_0}_{:= S_q(a)}$$

$$10\bar{a} + a_0 \xrightarrow{T_q} \bar{a} + \omega_q a_0 \overset{\textbf{Stack}}{=\!=\!=} 10 a_n a_{n-1} \ldots a_3 a_2 + (a_1 + \omega_q a_0)$$

$$\xrightarrow{T_q} a_n a_{n-1} \ldots a_3 a_2 + \omega_q (a_1 + \omega_q a_0)$$

$$\overset{\textbf{Stack}}{=\!=\!=} 10 a_n a_{n-1} \ldots a_3 + (a_2 + \omega_q a_1 + \omega_q^2 a_0) \qquad \cdots$$

Proof by induction on n $\quad (\textbf{Stack} \cdot T_q)^n(a) = S_q(a)$

Trimming for $q \longrightarrow$ Weighted Sum of Digits

$$S_{11}(2184) \; = \; 2 + (-1) \cdot 1 + (-1)^2 \cdot 8 + (-1)^3 \cdot 4 \quad [\omega_{11} = -1]$$

$$S_{7}(2184) \; = \; 2 + (-2) \cdot 1 + (-2)^2 \cdot 8 + (-2)^3 \cdot 4 \quad [\omega_{7} = -2]$$

$$S_{39}(2184) \; = \; 2 + 4 \cdot 1 + 4^2 \cdot 8 + 4^3 \cdot 4 \quad [\omega_{39} = 4]$$

$$S_{23}(2184) \; = \; 2 + 7 \cdot 1 + 7^2 \cdot 8 + 7^3 \cdot 4 \quad [\omega_{23} = 7]$$

$$S_{33}(2184) \; = \; 2 + 10 \cdot 1 + 10^2 \cdot 8 + 10^3 \cdot 4 \quad [\omega_{33} = 10]$$

Khare (1997): S_q via Modular Arithmetic $[\,\omega_q \equiv 10^{-1} \mod q\,]$

EO'S (2014): Trimming $\overset{\text{UNIFY!}}{\longrightarrow}$ Summing. [No Modular]

Comparison to Binomial Tests

$$a = \sum_{k=0}^{n} 10^k a_k \equiv \sum_{k=0}^{n} 1^k a_k =: B_9(a) \mod 9$$

$$a = \sum_{k=0}^{n} 10^k a_k \equiv \sum_{k=0}^{n} (10-q)^k a_k =: B_q(a) \mod q$$

Binomial Good for small weights $|10-q|$; bad otherwise.

$$a = \sum_{k=0}^{n} 10^k a_k \equiv \sum_{k=0}^{n} (-29)^k a_k =: B_{39}(a) \mod 39$$

Contrast $S_{39}(2184) = 2 + 4 \cdot 1 + 4^2 \cdot 8 + 4^3 \cdot 4 \quad [\omega_{39} = 4]$