INTERVAL VECTORS

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Abstract

The exploration of the profound and intrinsic cohesion between mathematics and music is certainly nothing new – it actually dates all the way back to Pythagoras (c. 570 BCE – c. 495 BCE).

Abstract

- However, the introduction of the ***dodecaphonic*** (***twelve-tone***) ***system*** developed by Arnold Schoenberg (1874 – 1951) has taken this study to entirely new levels, and has instituted such concepts as set theory, ordered sets, vectors, and various types of spaces as useful tools in music theory. In this paper we will look into one of these tools, namely the notion of interval vectors.

Basic Terminology

- Around 1908, the Viennese composer Arnold Schoenberg developed a system of pitch organization in which all twelve unique pitches were to be arranged into an ordered row. This row and the rows obtained from it by various basic operations were then used to generate entire pitch contents, giving rise to a method of composition now usually referred to as the ***dodecaphonic*** (***twelve-tone***) ***system*** or ***serialism***.
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Basic Terminology

This new system not only bolstered the existing ties between mathematics and music, but helped introduce some new ones as well. In fact, the field of ***musical set theory*** was developed by Hanson (1960) and Forte (1973) in an effort to categorize musical objects and describe their relationships in this new setting. For more information see Schuijer (2008) and Morris (1987).

Basic Terminology

Let us first review the basic terminology, starting with a notational convention. We will call the octave from middle C to the following B the ***standard octave***. If C denotes the middle C, we will use the convention $C = 0$.

Basic Terminology

- Then the remaining eleven unique pitches following C within the same octave can be numbered as $1, 2, 3, \ldots, 11$. So, with this convention,
- $C = 0, C^{\#} = 1, D = 2, D^{\#} = 3, E = 4, F = 5, F^{\#} = 6,\ G = 7, G^{\#} = 8, A = 9, A^{\#} = 10, B = 11.$

Basic Terminology

A ***pitch class*** is a set of all pitches that are a whole number of octaves apart, e.g., the pitch class of a pitch x consists of the x's in all octaves. Thus, the pitch class corresponding to a pitch x in the standard octave is the set

$$\{x_n : n \in \mathbb{Z}\}$$

where

$$x_n = x + 12n$$

Basic Terminology

Although there is no formal upper or lower limit to this sequence, by limitations of human ear, we can consider this to be a finite set. We will identify a pitch class by the numerical representation of its corresponding pitch in the standard octave, $0, 1, 2, \ldots, 11$. So, with our convention,

$$0 = \{C_n : n \in \mathbb{Z}\}$$

Basic Terminology

The distance between any two pitches is called a ***pitch interval*** (PI). Since we are interested in pitch classes rather than individual pitches, we will use the ***pitch interval class*** (PIC) rather than a pitch interval as the measure of distance. For any two pitch classes x and y,

$$PIC(x, y) = \min((x - y), (y - x)) \ (mod\ 12)$$

Basic Terminology

Clearly, there are six different pitch interval classes. If $PIC(x, y) = 1$, the distance between x and y is called a ***semitone***. In case $PIC(x, y) = 2$, the distance is called a ***whole-step***. For $PIC(x, y) = 3$, it is called a ***minor third***; for $PIC(x, y) = 4$, a ***major third***; for $PIC(x, y) = 5$, a ***prefect fifth***, and for $PIC(x, y) = 6$, a ***tritone***.

Basic Terminology

A ***pitch class set*** is simply an unordered collection of pitch classes (Rahn 1980, 27). More exactly, a pitch-class set is a numerical representation consisting of distinct integers (Forte 1973, 3), that is, any subset of the set $\{0, 1, , 2, \dots, 11\}$

We will use the { } notation for a pitch class set. Thus, the unordered set of pitch classes 3, 4, and 8 (corresponding in this case to $D^{\#}$, E, and $G^{\#}$) will be denoted as $\{3, 4, 8\}$. We have, obviously,

$$2^{12} - 1 = 4095$$

pitch class sets.

Basic Terminology

We use the notation $\langle \quad \rangle$ to denote an ordered set of pitch classes. For example, the ordered set $D^{\#}, E,$ and $G^{\#}$would be denoted as $\langle 3, 4, 8 \rangle$. An ordered set of n notes corresponds to an $n - tuple$.

The basic operations that may be performed on unordered pitch class sets are ***transposition, inversion, complementation,*** and ***multiplication***. Operations on ordered sequences of pitch classes are ***transposition, inversion, complementation, multiplication, retrograde,*** and ***rotation***.

Basic Terminology

Transposition is what we refer to as translation in mathematics. If x is a number representing a pitch class, its ***transposition by m semitones*** is written T_m and is defined as

$$T_m(x) = x + m \ (mod\ 12)$$

So, transposition of $B = 11$ by 3 semitones would be $11 + 3\ (mod\ 12) = 2 = D$.

Basic Terminology

We say a melody is ***inverted*** if the direction of its intervals is switched. The inversion operator is denoted by I. If the original interval between two consecutive pitch classes in a pitch class set goes up m semitones, the inversion goes down m semitones. Any note can be inverted by subtracting its value from 12: the inversion of k is $12 - k$ for $k = 0, \ldots, 11$, that is,

$$I(k) = 12 - k$$

$k = 0, \ldots, 11$. So for instance, the inversion of $\{3, 5, 8\}$ is $\{9, 7, 4\}$.

Basic Notation

The way it was defined, inversion corresponds to reflection around 12. Of course, we can invert (reflect) with respect to any fixed point in pitch class space. If x is a pitch class, the ***inversion with index number m*** is written I_m and is defined as

$$I_m(x) = m - x \ (mod\ 12)$$

So, inversion of $\{3, 5, 8\}$ with index number 10 is $\{7,5, 2\}$. Thus, $I = I_{12}$.

Basic Terminology

We can define a metric in a pitch class space as

$$dist(x, y) = PIC(x, y)$$

With this convention we note that transposition and inversion are isometries. In fact, this important result is referred to as the

The Central Postulate of Musical Set Theory: Transposition and inversion are isometries of a pitch class space, they preserve the intervallic structure of a set, and hence its musical character.

Basic Terminology

The *complement* of set S is the set consisting of all the pitch classes not contained in S (Forte 1973, 73–74). So for example, the complement of

$$S = \{0, 4, 7, 11\}$$

is the set

$$S^c = \{1, 2, 3, 5, 6, 8, 9, 10\}$$

The product of two pitch classes is the product of their pitch-class numbers modulo 12. So, the product of $C^\#$ and $A^\# = 2 \times 10 = 20\ (mod 12) = 8 = G^\#$

Since complementation and multiplication are not isometries of pitch-class space, they do not necessarily preserve the musical character of the objects they transform.

Basic Terminology

Retrograding an ordered sequence reverses the order of its elements. *Rotation* of an ordered sequence is equivalent to cyclic permutation.

We can, of course, consider compositions of these operations. For example, T_3I just means that we first invert the original set, and then transpose by three semitones. So,

$$T_3I\{1, 2, 7\} = T_3\{11, 10, 5\} = \{2, 1, 8\}$$

Interval Vectors

An ***interval vector*** (or ***interval-class vector***) for a pitch class set S, is a six-dimensional vector which summarizes the total interval content in S. The first component denotes the number of instances of $PIC(x, y) = 1$, the second component the number of instances of $PIC(x, y) = 2, \ldots,$ and the sixth component the number of instances $PIC(x, y) = 6$, for all x, y in the pitch class set S.

Interval Vectors

It is, essentially, a histogram of all of the interval classes which can be found in a pitch class set. Two sets that have the same interval content are called ***Z —-related sets***.

Interval Vectors

We will now examine the steps in calculating an interval vector. For this illustration we will use the pitch classes 8 6 0 1 and 5. We first arrange the pitch classes in ascending order. So now we will have

0 1 5 6 8

We then find the interval content by calculating the interval from each digit to the remaining digits, left to right:

Interval Vectors

Pitch Pairs x, y	$PIC(x, y)$
0, 1	1
0 , 5	5
0 , 6	6
0 , 8	4
1 , 5	4
1 , 6	5
1 , 8	5
5 , 6	1
5 , 8	3
6 , 8	2

Interval Vectors

Clearly, if we have n pitch classes, we will have

$$\frac{n(n-1)}{2}$$

such differences.

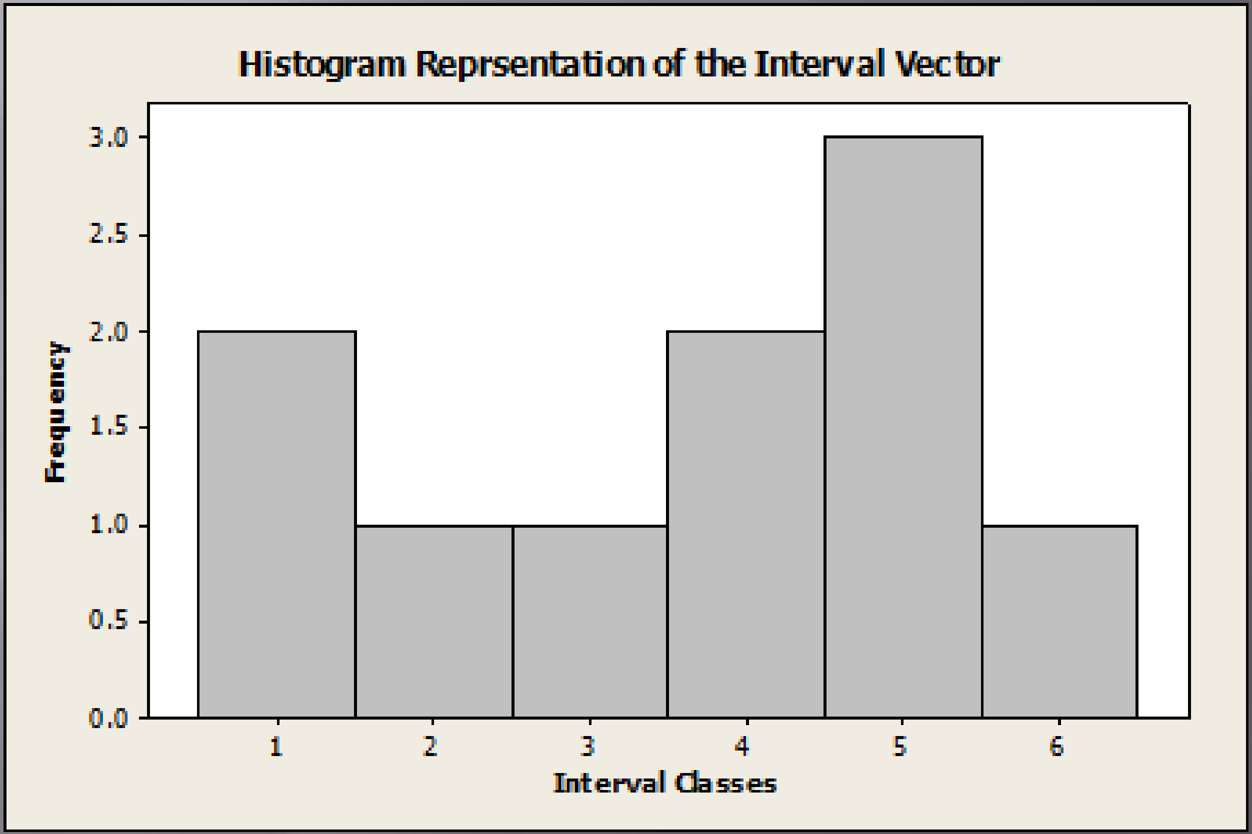
Interval Vectors

Finally, we count the total number of occurrences of each pitch interval class. In the above example, we have two intervals with $PIC(x, y) = 1$, one with $PIC(x, y) = 2$, one with $PIC(x, y) = 3$, two with $PIC(x, y) = 4$, three with $PIC(x, y) = 5$, and one with $PIC(x, y) = 6$. So the interval vector is

$$[2\ \ 1\ \ 1\ \ 2\ \ 3\ \ 1]$$

Of course, the sum of the components of this vector should equal $\frac{n(n-1)}{2}$.

Interval Vectors

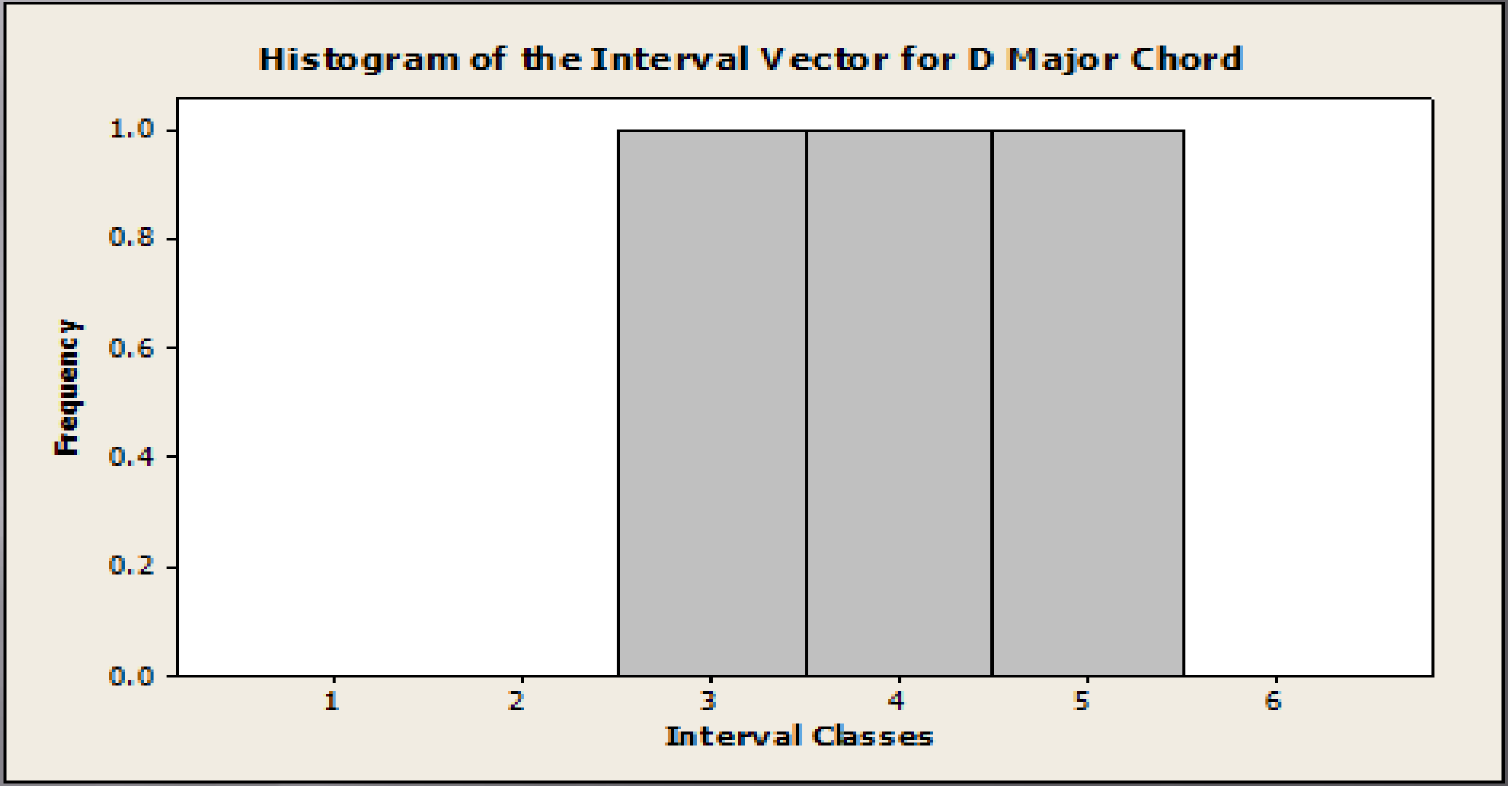


Interval Vectors

For example, a D major chord, which is represented by the pitch class set (2, 6, 9) will have the interval vector [0 0 1 1 1 0].

The E minor chord, which is represented by the pitch class set (4, 7, 11) will have the interval vector [0 0 1 1 1 0].

Interval Vectors



Applications of the Interval Vector

The first use of the interval vector is to tabulate the intervals in a pitch class set. Since these intervals, taken together, give a set its characteristic sound, sets that have similarities in their vectors will be more alike than those that do not. In fact, one could establish numeric measurements of similarity based on the interval vector. I will talk about this in another paper.

Applications of the Interval Vector

The interval vector also tells us about what will happen when we transpose a set:

Suppose the entry in the j^{th} *position for* $j = 1, \dots, 5,$ *of the interval vector is* k. *Then the original pitch class set and its transposition by* j *or* $12 - j$ *units will have* k *pitches in common. If* $j = 6$, *then there will be* $2k$ *pitches in common.* This is known as the ***Common Tone Theorem***.

Applications of the Interval Vector

For example, take the pitch class set $(1, 2, 7, 8)$. Its interval vector is [2 0 0 0 2 2]. This means $T_1(1, 2, 7, 8)$ and $T_{11}(1, 2, 7, 8)$ will both have two pitches in common with $(1, 2, 7, 8)$. Of course this can easily be verified since

$$T_1(1, 2, 7, 8) = (2, 3, 8, 9)$$

and

$$T_{11}(1, 2, 7, 8) = (0, 1, 6, 7)$$

Applications of the Interval Vector

Again we can easily verify that

$$T_2(1,2,7,8), T_{10}(1,2,7,8), T_3(1,2,7,8), T_9(1,2,7,8), T_4(1,2,7,8), T_8\,(1,2,7,8),$$

$$T_5(1,2,7,8), T_7(1,2,7,8)$$

have no pitches in common with $(1,2,7,8)$.

Applications of the Interval Vector

Note that

$$T_6(1, 2, 7, 8) = (7, 8, 1, 2)$$

has exactly four pitches in common with $(1, 2, 7, 8)$.

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