When Is a Cube Like a Tetrahedron?

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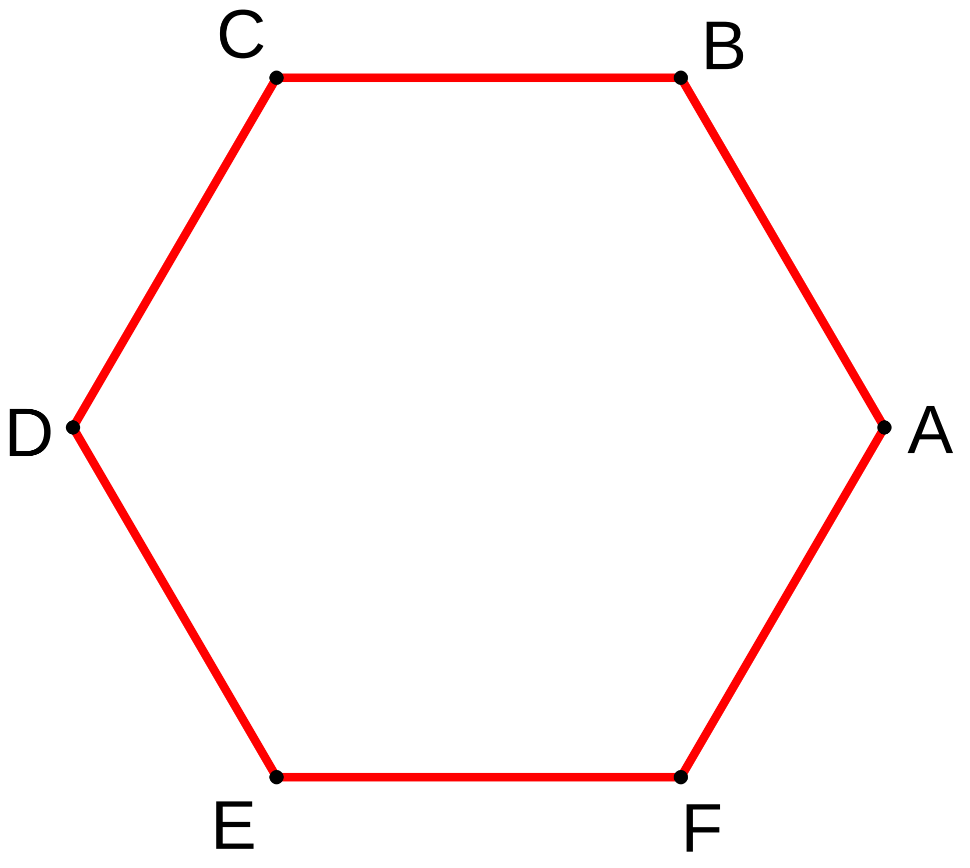
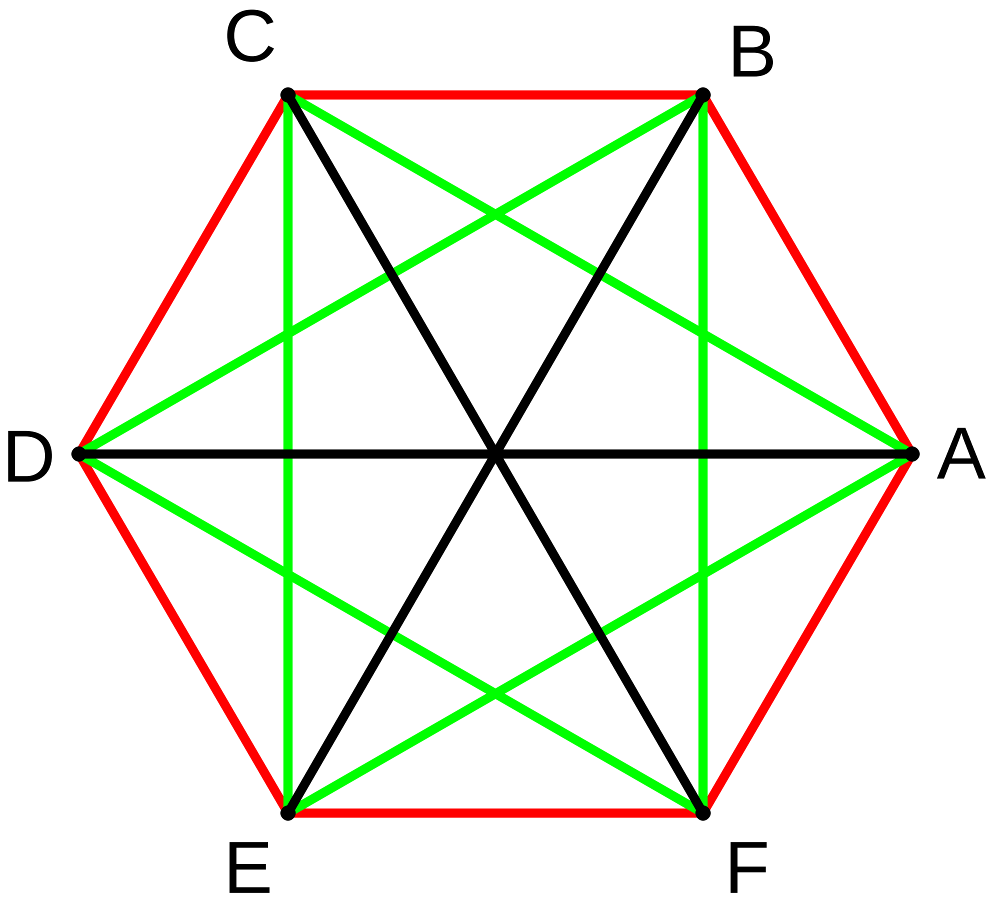

C
B
D
A
E
F



C
B
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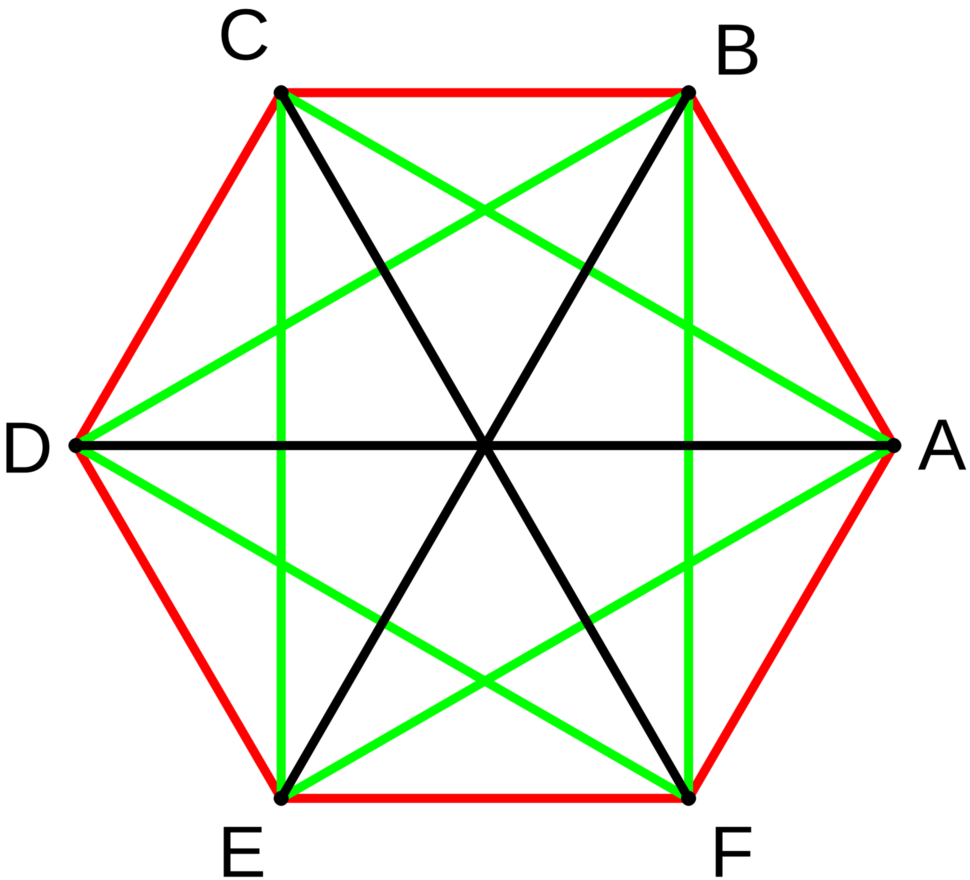
C
B
D
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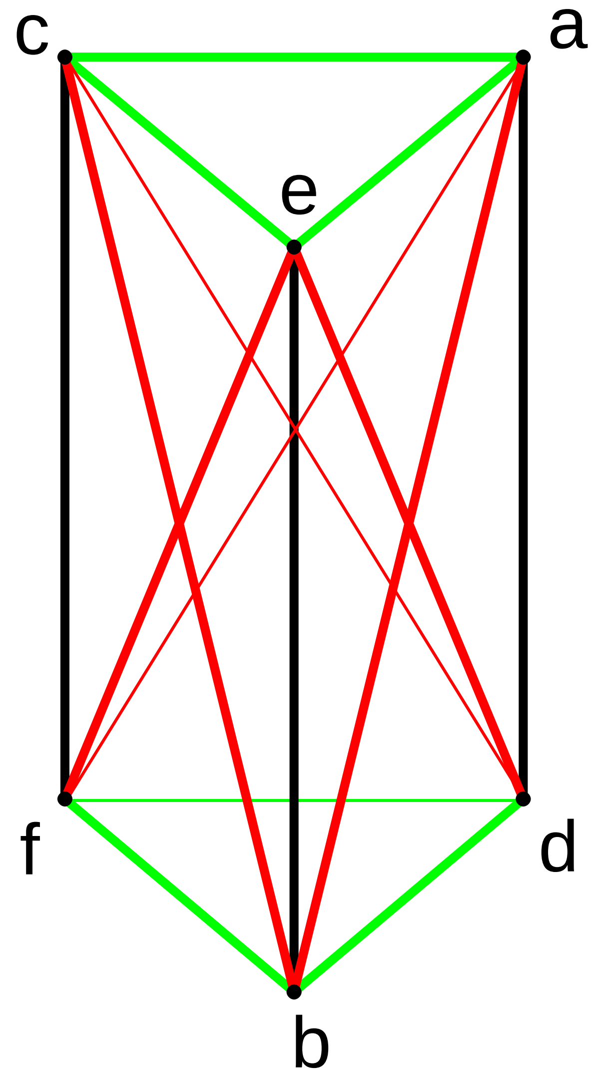
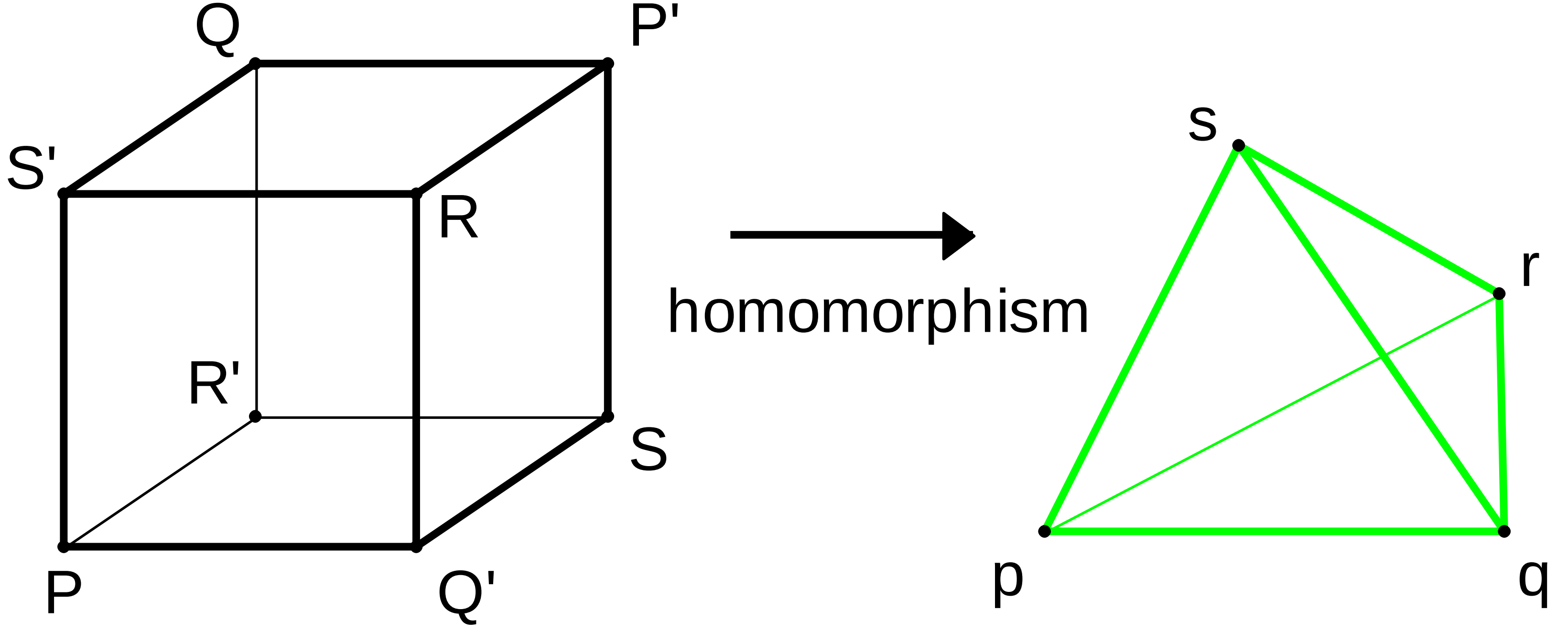
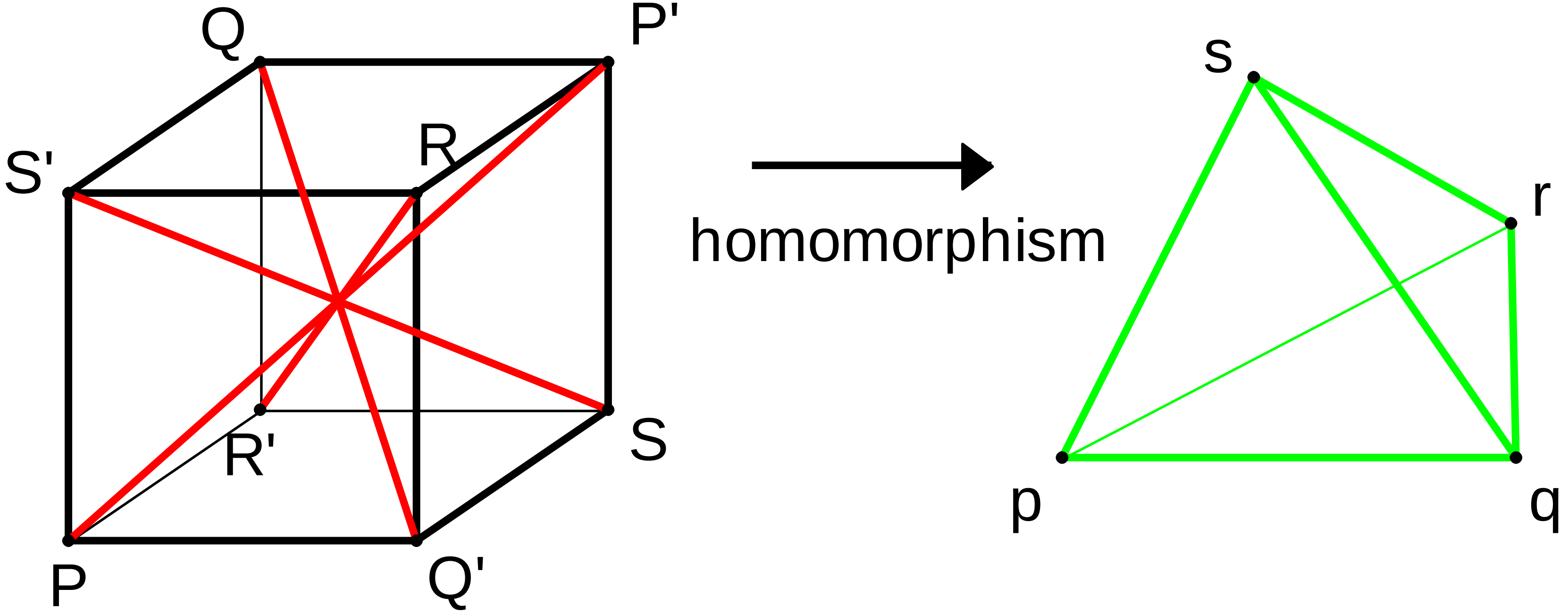
F

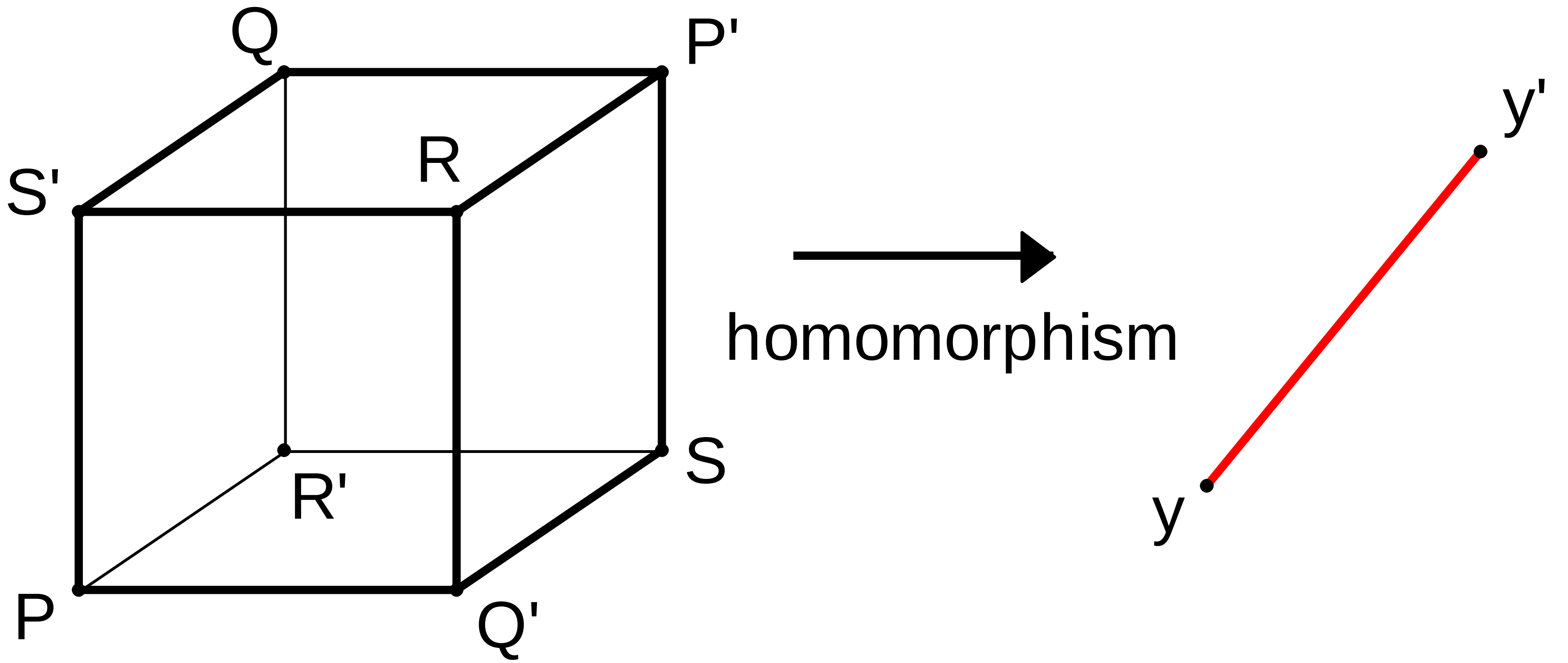

c
a
e
f
d
b

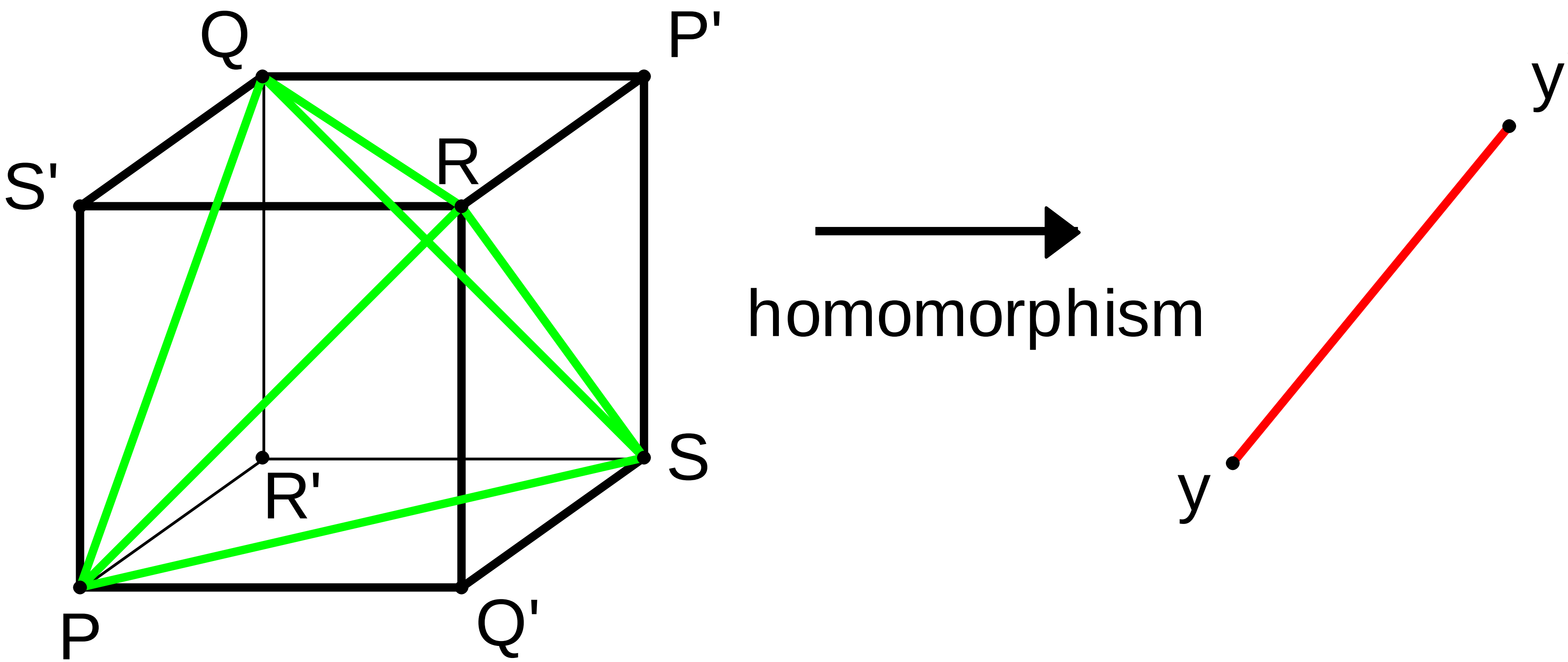
- **Definition (S,C)** is a *space* iff **S** is a set and **C** is a function with domain **S×S** such that
  i) **C(x,y) = C(y,x)** and
  ii) **C(x,x) = C(y,z)** iff **y = z**.

- **Definition**. A *homomorphism* from **(S,C)** to **(T,D)** is a function **h: S→T** so that for all **p, q, r, s** ∈ **S** if **C(p,q) = C(r,s)**, then **D(h(p),h(q)) = D(h(r),h(s))**.
  The *coset* of **p** ∈ **S** under **h** is H_p = {x: h(x) = h(p)}.


Q
P'
S'
R
R'
S
P
Q'
homomorphism
s
r
p
q


Q
P'
S'
R
R'
S
P
Q'
homomorphism
s
r
p
q


Q
P'
R
S'
S
R'
P
Q'
homomorphism
y'
y

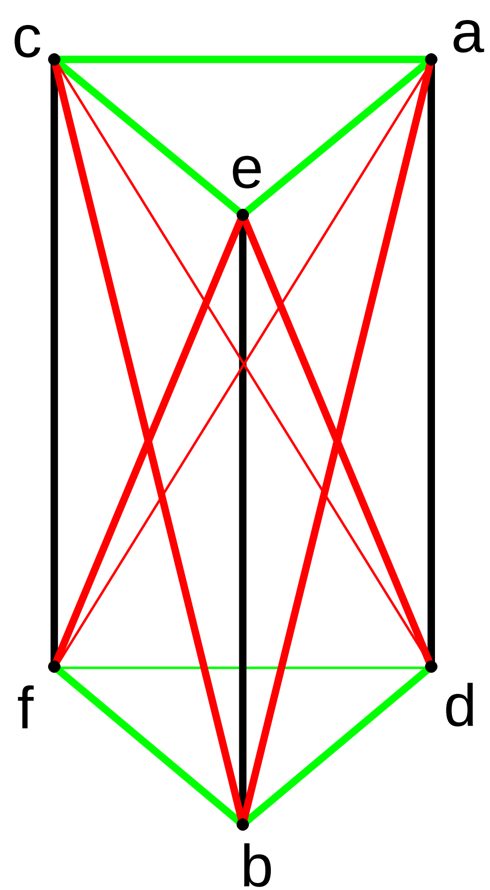

Q
P'
S'
R
S
R'
P
Q'
homomorphism
y'
y

- **Theorem 1.** If **h: S→T** is a homomorphism, **p, q, r, s ∈ S**, **C(p,q) = C(r,s)** and **q ∈ H_p**, then **s ∈ H_r**.

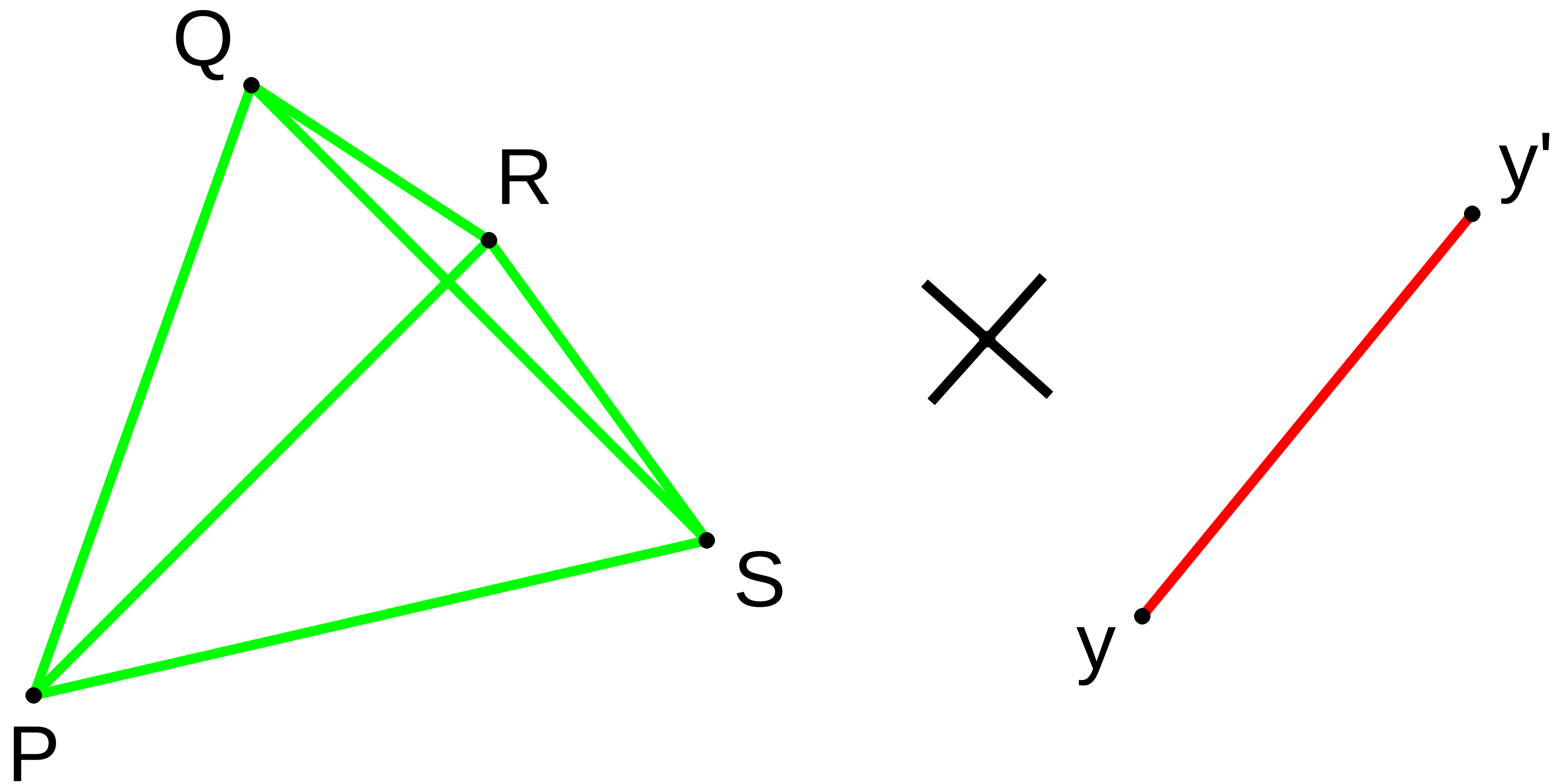
- **Definition**. A subset **S** of a space **X** is a *subspace* iff for all **p, q, r** ∈ **S** and **s** ∈ **X** if **C(p,q) = C(r,s)**, then **s** ∈ **S**.

- **Corollary**. A coset of a homomorphism is a subspace.

Subspaces?

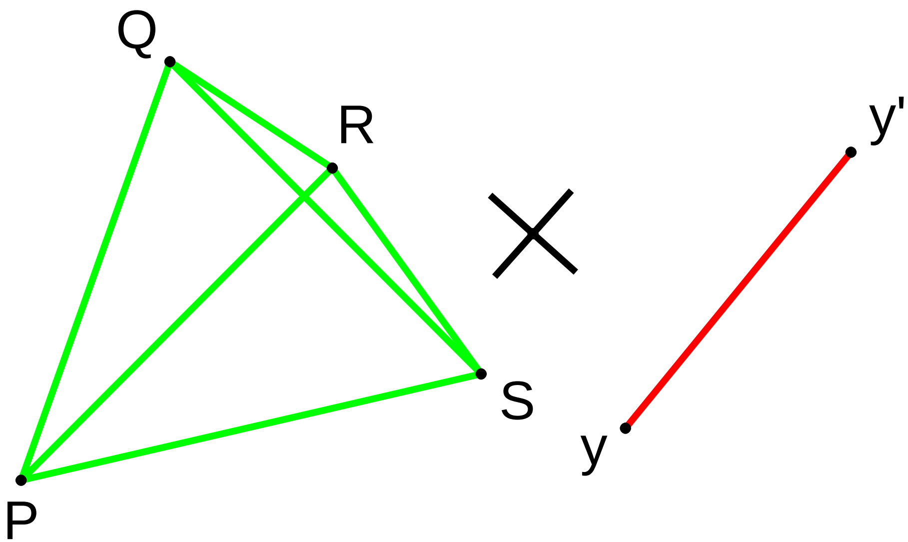


Direct Product



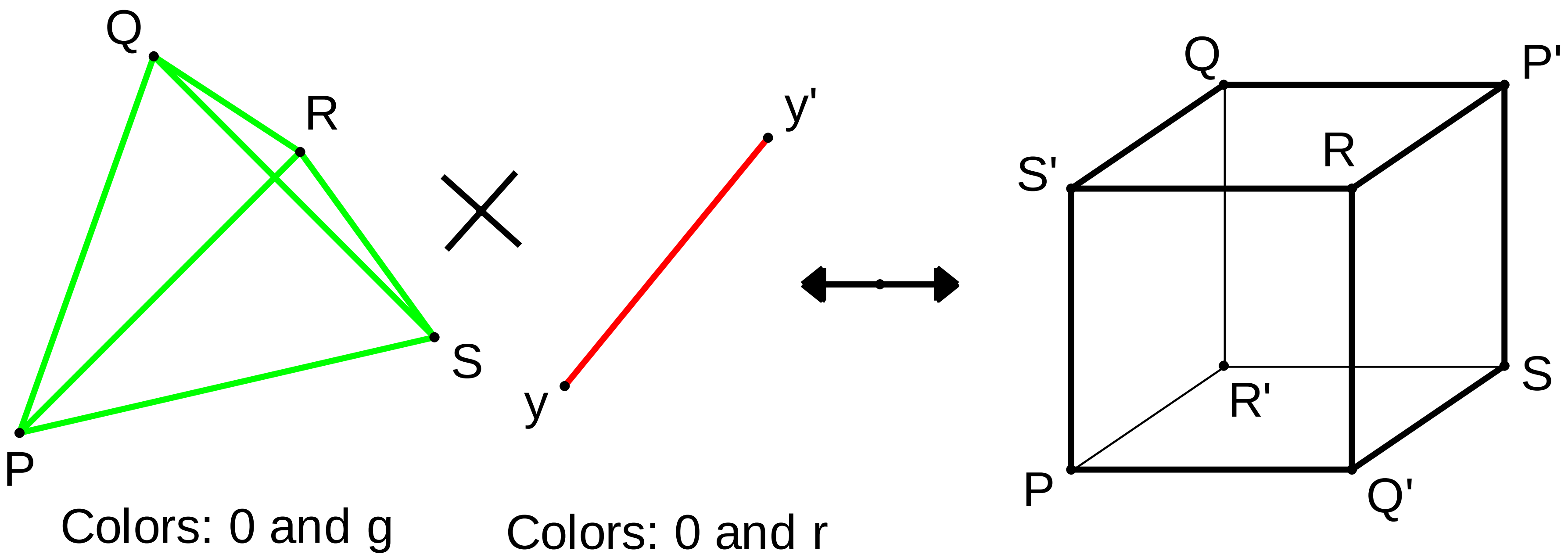
Direct Product

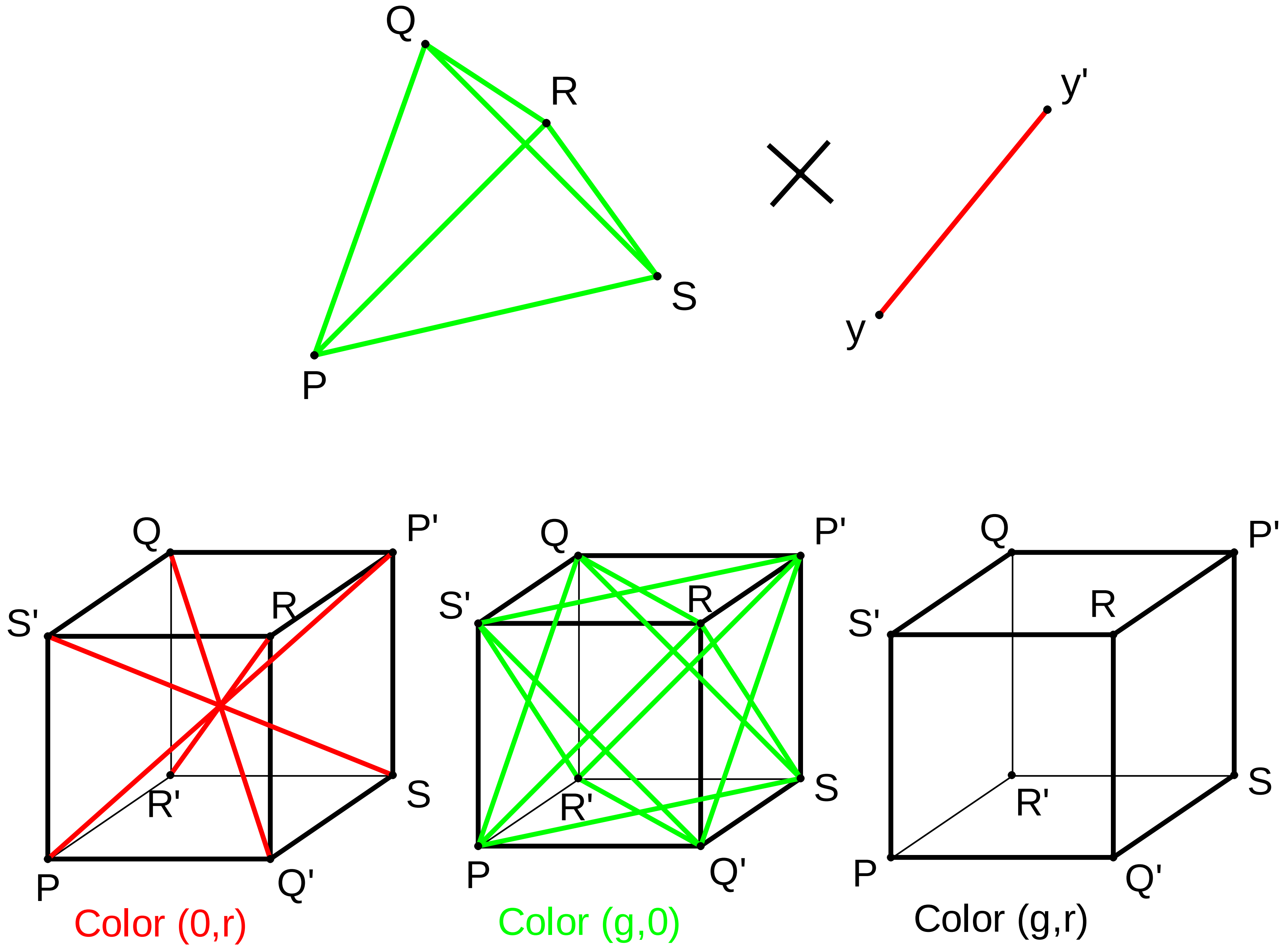
- **Definition**. Given **(X, C)** and **(Y, D)**, define **(X×Y, C×D)** on **X×Y** with color function **C×D((p, q), (r, s)) = (C(p,r), D(q,s))**.



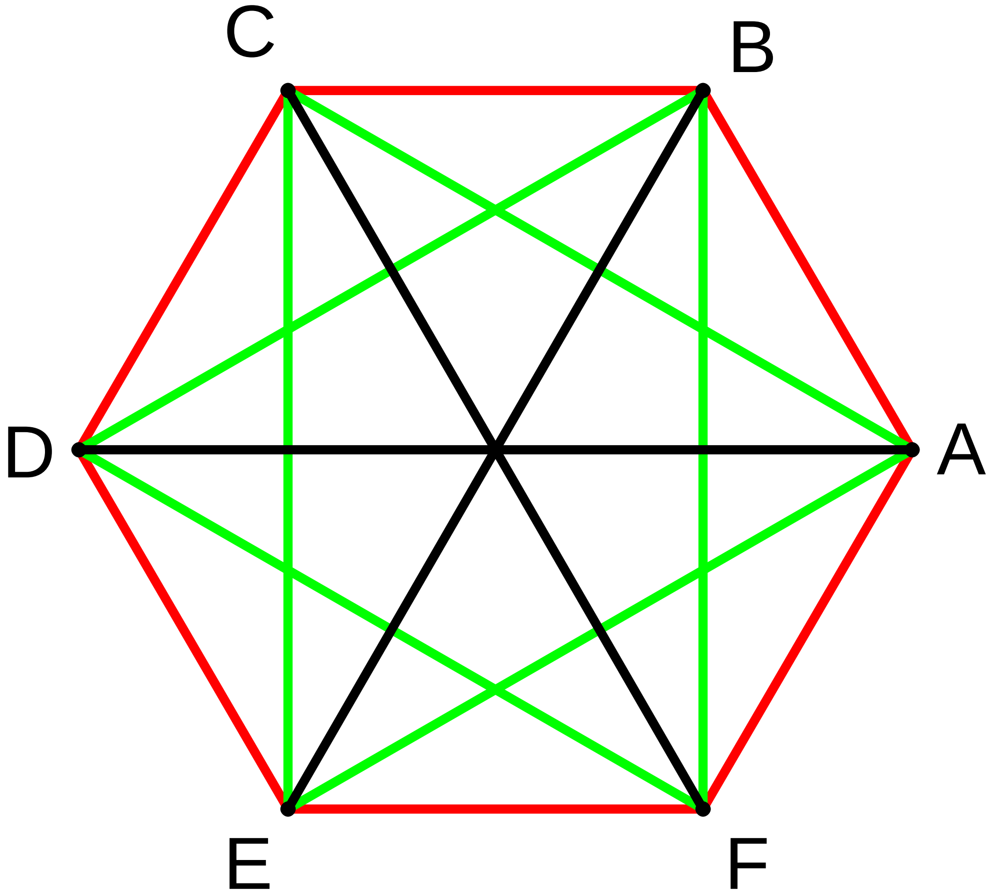
Colors: 0 and g

Colors: 0 and r

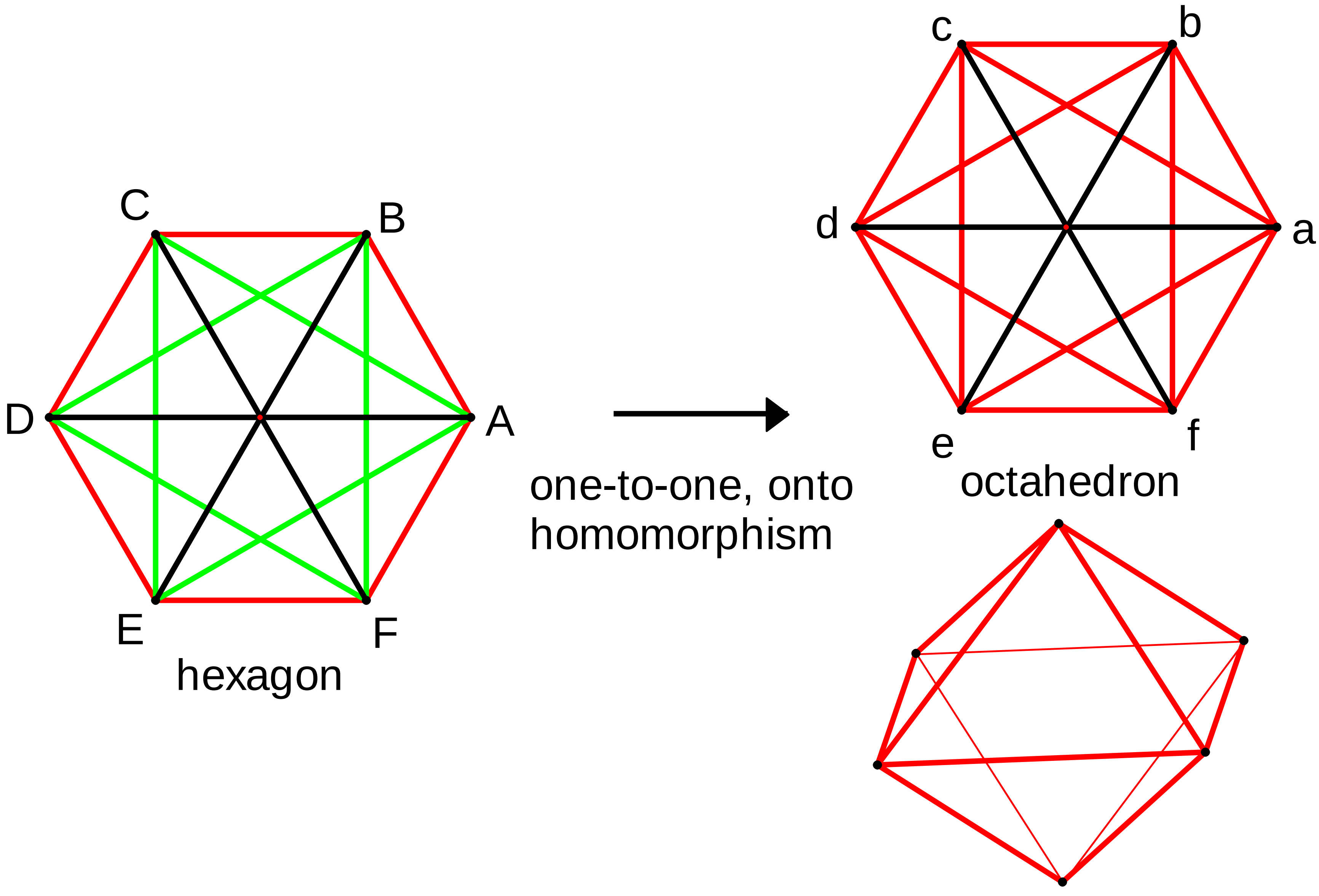

Q
R
S
P
Colors: 0 and g
y'
y
Colors: 0 and r
Q
P'
S'
R
S
R'
P
Q'



Q
R
S
P
y'
y
Q
P'
S'
R
R'
S
P
Q'
Color (0,r)
Q
P'
S'
R
R'
S
P
Q'
Color (g,0)
Q
P'
S'
R
R'
S
P
Q'
Color (g,r)


C
B
D
A
E
F

- **Definition.** If **h: X→Y** is a one-to-one onto homomorphism, **(Y,D)** *contains* **(X,C)**. If each space contains the other, the spaces are *isomorphic*.


C
B
D
A
E
F
hexagon
one-to-one, onto
homomorphism
c
b
d
a
e
f
octahedron

Definitions. A bijection σ on **(X, C)** preserving colors is an <u>*isometry*</u>.
(For all **p, q ∈ X**, **C(p, q) = C(σ(p), σ(q))**.)

I(X) is the group of isometries of **X**.

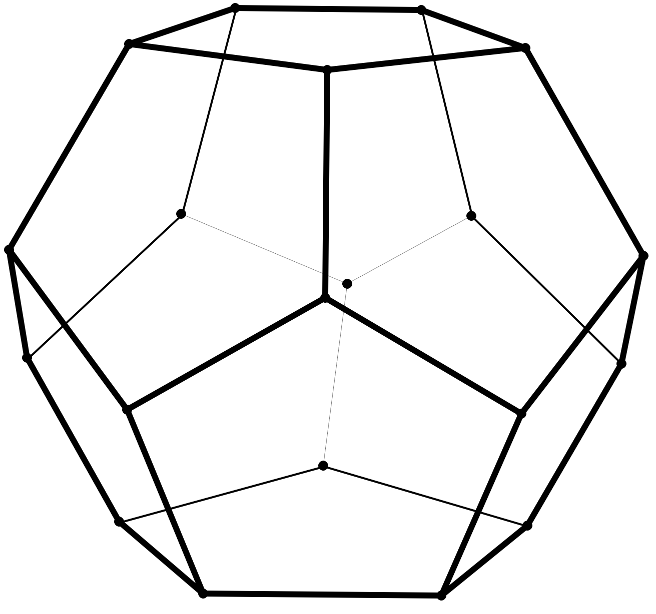
A space **(X, C)** is <u>*one-point homogeneous*</u> iff **I(X)** is <u>*transitive*</u>; that is, for all **a, b ∈ X**, there is **σ ∈ G** such that **σ(a) = b**.

- **Theorem 2**. **I(X×Y)** is isomorphic to **I(X)×I(Y)**. If **X** and **Y** are one-point homogeneous, so is **X×Y**.

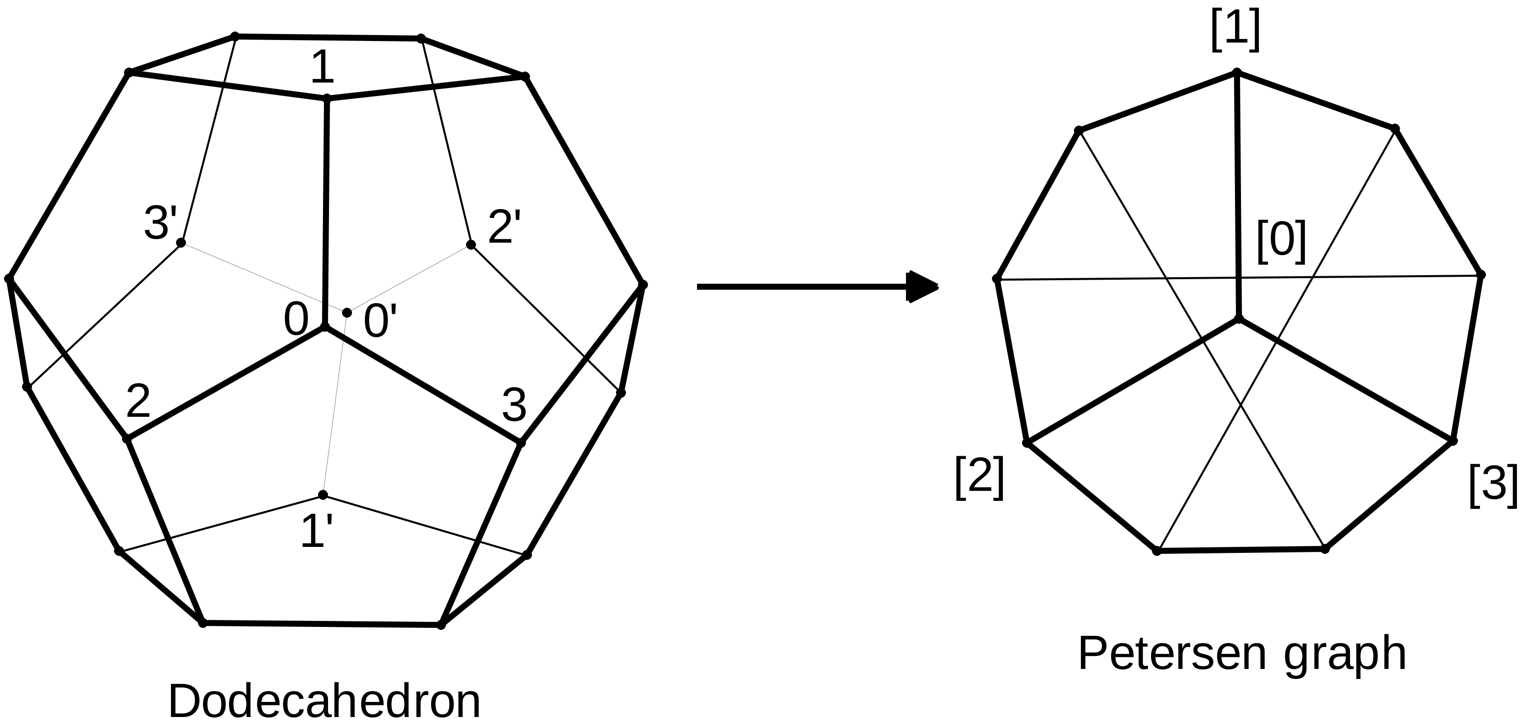
- **Theorem 3**. If **(X, C)** is a finite one-point homogeneous space and **Y** is a subspace,

i) (**LaGrange's Theorem**) **|Y|** divides **|X|**,
ii) **X** can be partitioned into closed subspaces “isometric” to **Y**.

Homomorphic Image?



Dodecahedron

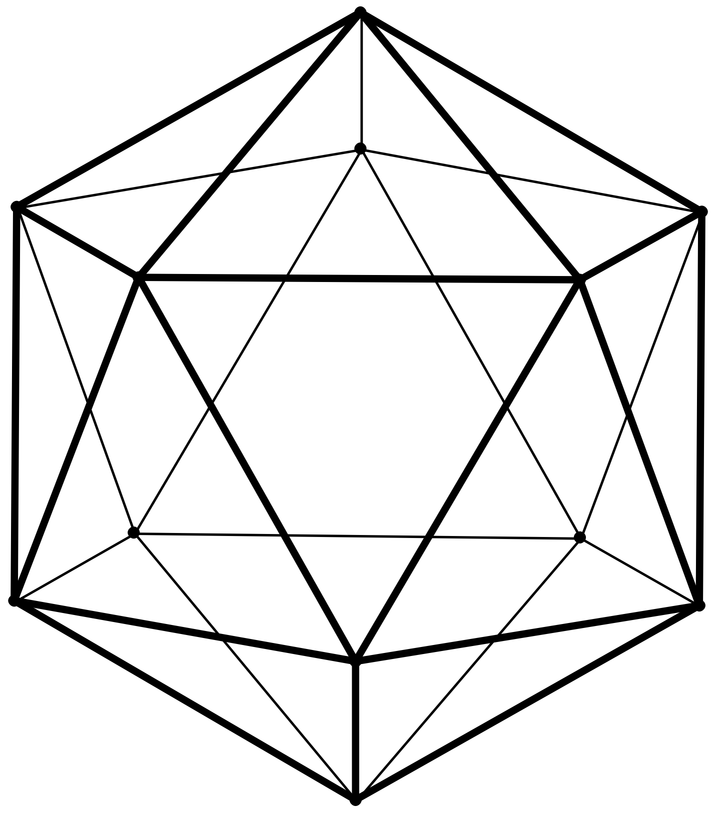


1
3'
2'
0
0'
2
3
1'
Dodecahedron
[1]
[0]
[2]
[3]
Petersen graph

- **I(Dodecahedron)** $\cong$ $\mathbf{A_5 \times Z_2}$ with 120 elements and **I(Petersen)** $\cong$ $\mathbf{S_5}$, also with 120 elements.

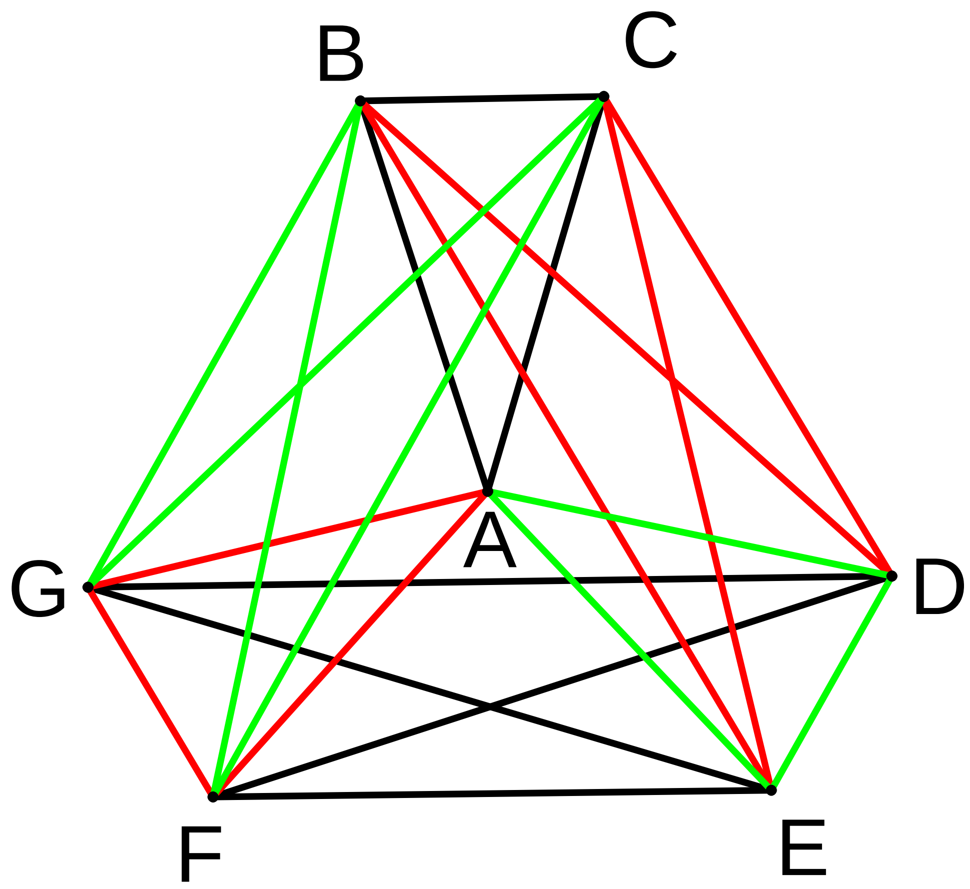
- **Theorem 4.** If **h: X→Y** is a homomorphism onto **Y**, there is a homomorphism of **I(X)** into **I(Y)**.

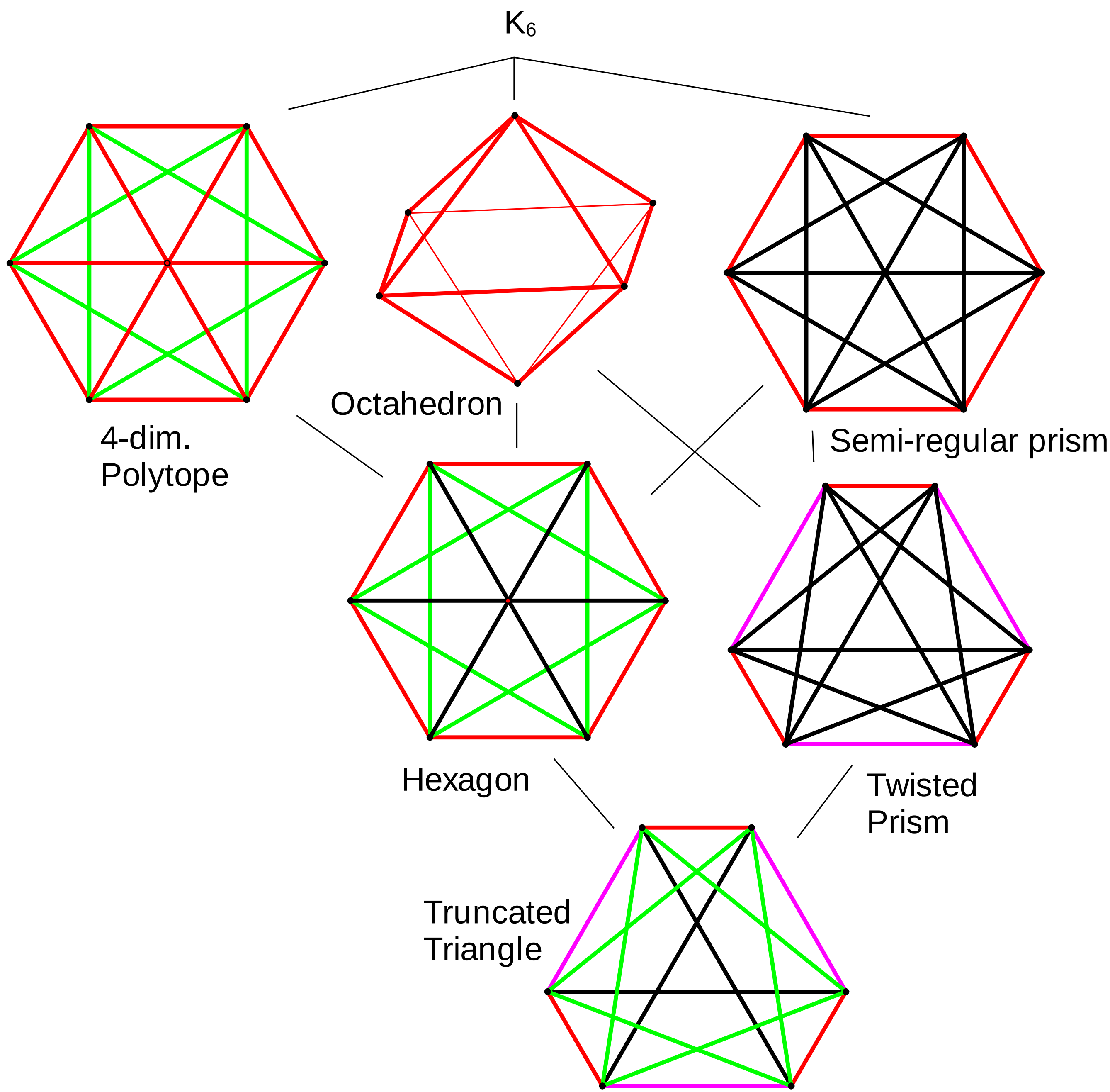
Homomorphic Image?



Icosahedron

Non-homogeneous Space




K_6
4-dim.
Polytope
Octahedron
Semi-regular prism
Hexagon
Twisted
Prism
Truncated
Triangle

- **Theorem 5.** Let **Z** be the homomorphic image of the one-point homogeneous space **X** and the cosets all be isometric to the subspace **Y**. If **X** contains **Y×Z**, then the natural homomorphism from **I(X)** to **I(Z)** is onto.