Some different applications of logarithms
Brian Heinold
Mount St. Mary’s University
1. The solution to $b^x = c$ is $x = \log_b(c) = \frac{\ln c}{\ln b}$

2. $\log(xy) = \log(x) + \log(y)$

That is, a multiplicative change in the input corresponds to an additive change in the output.

For example:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 12 \log_{10}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>100</td>
<td>24</td>
</tr>
<tr>
<td>1000</td>
<td>36</td>
</tr>
<tr>
<td>10000</td>
<td>48</td>
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How many rolls will it take, on average, until all the dice are the same? **About 11**

What if we had 10 dice?
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What if we had 10 dice? **About 15.3**

A multiplicative change of 10 in the number of dice corresponds to an additive change of roughly 13 in the number of rolls.
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Rolling dice

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So roughly, if we start with $N$ dice, solving $N \left(\frac{5}{6}\right)^k = 1$ will tell us how long it takes to get down to one die remaining.
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Increasing $N$ by a factor of 10 corresponds to an increase of

$$\log_{6/5}(10N) - \log_{6/5}(N) = \log_{6/5}(10) \approx 12.6$$

rolls.
Benford’s Law

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- 80 to 90 is an increase of about 12%, while 100 to 200 requires a doubling
Like a slide rule

Look at how much bigger on a log scale the gap from 1 to 2 is versus the gap from 8 to 9.
Benford’s Law

- The formula below gives the probability of starting with digit $d$:

$$ P(d) = \log_{10}(d + 1) - \log_{10}(d) = \log_{10} \left( 1 + \frac{1}{d} \right) $$

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- Nice Radiolab episode on Benford’s:
  http://www.radiolab.org/2009/nov/30/
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Weber-Fechner law: the amount of perception is proportional to $\ln\left(\frac{S}{S_0}\right)$, where $S$ is the amount of stimulus and $S_0$ is the smallest stimulus that is perceivable.
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The birthday problem

How many people need to be in a room for there to be a 50/50 chance that two people share a birthday?
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- With $k$ people in a room, the probability of no shared birthdays is

$$\frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdots \frac{362 - (k - 1)}{365}$$

$$= \left(1 - \frac{1}{365}\right) \left(1 - \frac{2}{365}\right) \left(1 - \frac{3}{365}\right) \cdots \left(1 - \frac{k - 1}{365}\right)$$

$$\approx e^{-1/365} e^{-2/365} \cdots e^{-(k-1)/365}$$

$$= e^{-k(k-1)/(2 \cdot 365)}$$

$$\approx e^{-k^2/(2 \cdot 365)}$$
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- Invert this to get the number of people needed for there to be a probability $p$ of a repeat:

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Another Example: Hash function collisions
Other places logs show up

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Discrete logarithm: $b^x = c$ in a group. For example, if $p$ is prime, solve $3^x \equiv 50 \pmod{71}$. Useful in cryptography.
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- Hand calculations before calculators
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Binary trees

Number of levels is roughly $\log_2 n$, where $n$ is the number of elements.
Logs are useful if you need to multiply a lot of really small numbers together.

For instance, multiplying 500 numbers between .01 and .1 would result in 0 on a computer (underflow).

Say we need to compare \( \prod_{i=1}^{500} p_i \) with \( \prod_{i=1}^{500} q_i \), where \( p_i, q_i < 1. \)

We can just compare the logs of the products, using the fact that \( \log(\prod_{i=1}^{500} p_i) = 500 \sum_{i=1}^{500} \log(p_i) \).

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Professor Weiss is a three-time *Jeopardy!* champion. Let’s say the probability that he beats you in a game of *Jeopardy!* is 0.9999.

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\[ .9999^{200} \]

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\[ 1 - .9999^x = .9 \]

\[ x = \frac{\ln(.9)}{\ln(.9999)} \approx 1054 \]
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   \[1 - 0.9999^x = 0.9 \rightarrow x = \frac{\ln(0.9)}{\ln(0.9999)} \approx 1054\]
Iterated function systems

Figure on the left is colored according to whether a point is hit or not.

Figure on the right is colored according to the log of the number of times the point was hit.

Some points are hit rarely, while others are hit thousands of times. Take the log of the number times a point was hit and use that for shading.
Why is $\int_1^x \frac{1}{t} \, dt = \ln x$?
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Because it’s defined that way?

Here’s a modernization of the approach first taken by Mercator and St. Vincent in the 1600s.
\[ \int_1^x \frac{1}{t} \, dt \] is the area under \( y = 1/t \) from \( t = 1 \) to \( t = x \).
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A multiplicative change in \( x \) corresponds to an additive change in the area.
What base?

This leads to

\[ \int_{1}^{x} \frac{1}{t} \, dt = \log x. \]

But what is the base?
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\[ \int_{1}^{x} \frac{1}{t} \, dt = \log x. \]

But what is the base?

We know the base is \( e \).

But why not something else, like base 7 or base 443.18? 
Why base $e$

Say we want $\int_1^{32} \frac{1}{x} \, dx$.

Suppose instead of powers of 2, we use something smaller, like powers of $r \in (1, 2)$. The smaller rectangles will fit the area more closely.

How many rectangles will there be?

Answer: Find the largest power of $r$ less than 32. In other words, solve $r^x = 32$. We get $x = \frac{\log(32)}{\log(r)}$. 
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Suppose instead of powers of 2, we use something smaller, like powers of \( r \in (1, 2) \).

The smaller rectangles will fit the area more closely.

How many rectangles will there be?

Each has area = \( r-1 \)
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The area is then \( \frac{\log(32)}{\log r} (r - 1) \).
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The area is then

$$\frac{\log(32)}{\log(1 + \frac{1}{n})} (1 + \frac{1}{n} - 1)$$

$$= \frac{\log(32)}{n \log(1 + \frac{1}{n})}$$

$$= \frac{\log(32)}{\log(1 + \frac{1}{n})^n}$$

$$= \log(1 + \frac{1}{n})^n(32)$$

As $n \to \infty$, this becomes $\log_e(32)$. 