Some different applications of logarithms

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Important facts about logarithms

1. The solution to $b^x = c$ is $x = \log_b(c) = \dfrac{\ln c}{\ln b}$

2. $\log(xy) = \log(x) + \log(y)$

   That is, a multiplicative change in the input corresponds to an additive change in the output.

   For example:

x	$y = 12\log_{10}(x)$
1	0
10	12
100	24
1000	36
10000	48

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- A multiplicative change of 10 in the number of dice corresponds to an additive change of roughly 13 in the number of rolls.

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- Increasing N by a factor of 10 corresponds to an increase of

$$\log_{6/5}(10N) - \log_{6/5}(N) = \log_{6/5}(10) \approx 12.6 \text{ rolls.}$$

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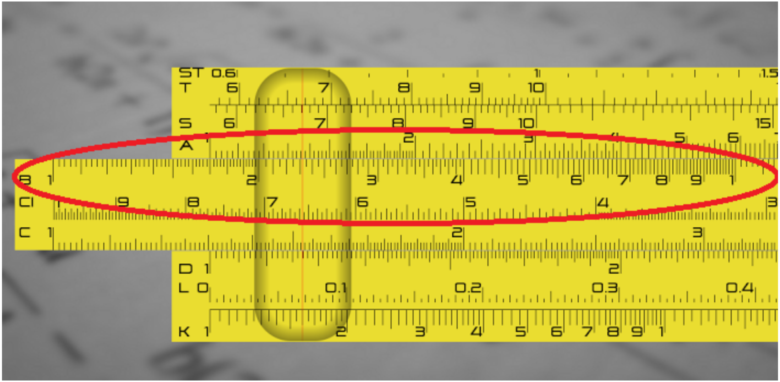
- 80 to 90 is an increase of about 12%, while 100 to 200 requires a doubling

Like a slide rule

Look at how much bigger on a log scale the gap from 1 to 2 is versus the gap from 8 to 9.



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- The formula below gives the probability of starting with digit d:

$$P(d) = \log_{10}(d+1) - \log_{10}(d) = \log_{10}\left(1 + \frac{1}{d}\right)$$

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2	17.6
3	12.5
4	9.7
5	7.9

6	6.7
7	5.8
8	5.1
9	4.6

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- Nice Radiolab episode on Benford's: `http://www.radiolab.org/2009/nov/30/`

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- Our perception of sound loudness, brightness, and touch sensitivity is also logarithmic.
- Weber-Fechner law: the amount of perception is proportional to $\ln\left(\frac{S}{S_0}\right)$, where S is the amount of stimulus and S_0 is the smallest stimulus that is perceivable.

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- With k people in a room, the probability of no shared birthdays is

$$
\begin{aligned}
& \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \dots \cdot \frac{362-(k-1)}{365} \\
=& \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right) \dots \left(1 - \frac{k-1}{365}\right) \\
\approx & e^{-1/365} e^{-2/365} \dots e^{-(k-1)/365} \\
=& e^{-k(k-1)/(2 \cdot 365)} \\
\approx & e^{-k^2/(2 \cdot 365)}
\end{aligned}
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- Another Example: Hash function collisions

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- Hand calculations before calculators

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- Number of levels is roughly $\log_2 n$, where n is the number of elements

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- We can just compare the logs of the products, using the fact that
$$\log\left(\prod_{i=1}^{500} p_i\right) = \sum_{i=1}^{500} \log(p_i).$$
- Underflow is not a problem for this sum.

Basic probability example

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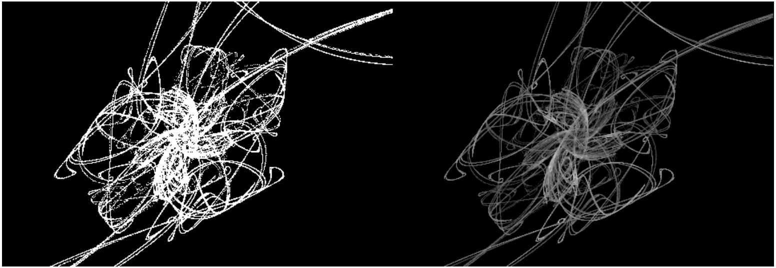
   $$1 - .9999^x = .9 \to x = \frac{\ln(.9)}{\ln(.9999)} \approx 1054$$

Iterated function systems

Figure on the left is colored according to whether a point is hit or not.

Figure on the right is colored according to the log of the number of times the point was hit.



Some points are hit rarely, while others are hit thousands of times. Take the log of the number times a point was hit and use that for shading.

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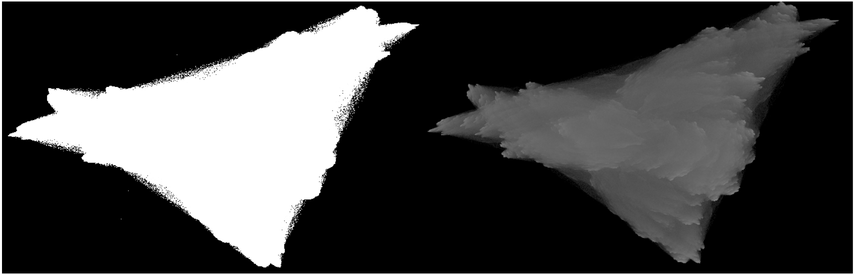

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Here's a modernization of the approach first taken by Mercator and St. Vincent in the 1600s.

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$\int_1^x \frac{1}{t}\,dt$ is the area under $y = 1/t$ from $t = 1$ to $t = x$.

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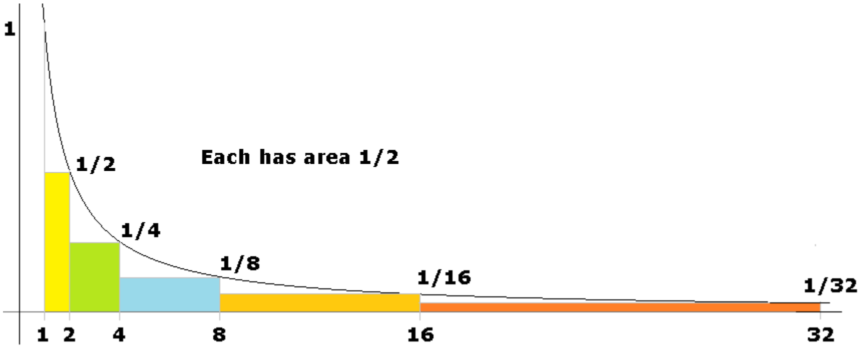

A multiplicative change in x corresponds to an additive change in the area.

What base?

This leads to

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We know the base is e.

But why not something else, like base 7 or base 443.18?

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Suppose instead of powers of 2, we use something smaller, like powers of $r \in (1, 2)$.
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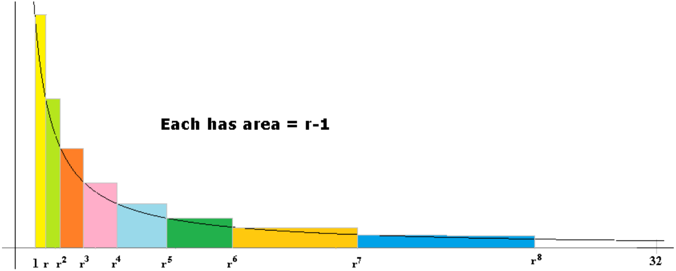

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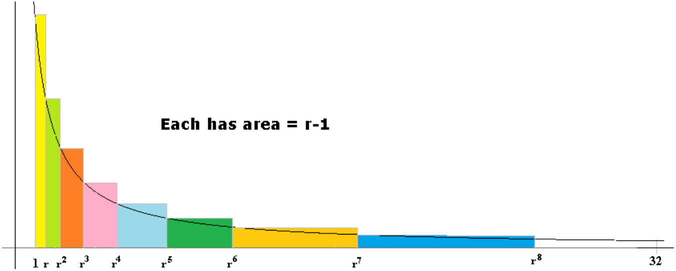

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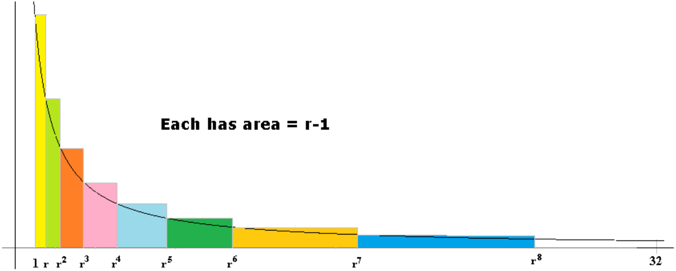

How many rectangles will there be?
Answer: Find the largest power of r less than 32.
In other words, solve $r^x = 32$. We get $x = \frac{\log(32)}{\log r}$.

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The area is then $\frac{\log(32)}{\log r}(r-1)$.

Suppose $r = 1 + \frac{1}{n}$ for some small value of n.

The area is then

$$\begin{aligned}
&\frac{\log(32)}{\log(1+\frac{1}{n})}(1+\frac{1}{n}-1) \\
=&\frac{\log(32)}{n\log(1+\frac{1}{n})} \\
=&\frac{\log(32)}{\log(1+\frac{1}{n})^n} \\
=&\log_{(1+\frac{1}{n})^n}(32)
\end{aligned}$$

As $n \to \infty$, this becomes $\log_e(32)$.