From points of inflection to bones of contention: the birth of block designs, normed algebras, and finite geometries

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What To Expect

1835: Plücker, inflection points on cubics, and the $(9, 3, 1)$ block design

1843-45: Graves, Cayley, the octonions, and seven triples

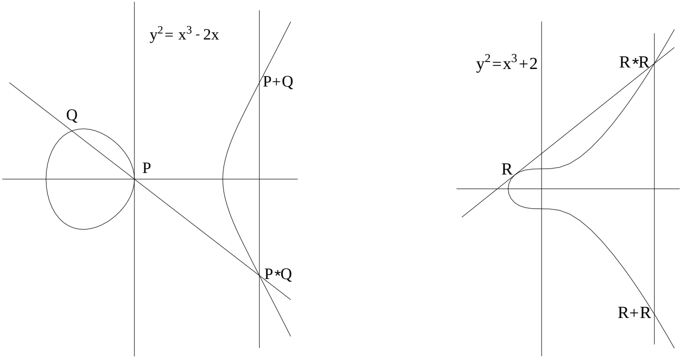
1844-47: Woolhouse, Kirkman, and the $(7, 3, 1)$ design

1853: Steiner and the "first" triple systems

1892: Fano, the "first" finite geometry, and $(7, 3, 1)$

2013: Conclusions – not quite the end

Elliptic curves



- An *elliptic curve* is the set of all points (x, y) where $y^2 = g(x)$ where g is a cubic polynomial with three distinct roots.
- A *point of inflection* on an elliptic curve is a point (x, y) on the curve where y'' is defined and changes sign.

Plücker

Julius Plücker (1801-1868)

1835: Plücker's discovery

The first finite geometry

The nine points of inflection on an elliptic curve:

8	1	6
3	5	7
4	9	2

The nine-point affine plane $AG(2,3)$

$(9,3,1)$: the first "Steiner" triple system

three rows:	$\{1,6,8\},\{2,4,9\},\{3,5,7\}$
three columns:	$\{1,5,9\},\{2,6,7\},\{3,4,8\}$
three main diagonals:	$\{1,4,7\},\{2,5,8\},\{3,6,9\}$
three off diagonals:	$\{1,2,3\},\{4,5,6\},\{7,8,9\}$

9 points, 3 points to a line, each pair of points on exactly 1 line

Normed Algebras

Definition

A *normed algebra* $\mathbb{A}$ is an n-dimensional vector space over the real numbers $\mathbb{R}$ such that

- $\alpha(xy) = (\alpha x)y = x(\alpha y)$, for all $\alpha \in \mathbb{R}, x, y \in \mathbb{A}$
- $x(y+z) = xy + xz$, $(y+z)x = yx + zx$, $\forall x, y, z \in \mathbb{A}$
- There exists a function $N : \mathbb{A} \to \mathbb{R}$ such that $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{A}$.

Examples

- The real numbers $\mathbb{R}$ form a one-dimensional normed algebra with $N(x) = x^2$.
- The complex numbers $\mathbb{C}$ form a two-dimensional normed algebra with $N(x+iy) = (x+iy)(x-iy)$.
- There are two others.

William Rowan Hamilton (1805-1865)

Graves

John Thomas Graves (1806-1870)

1843-44

October

Hamilton writes to Graves describing his four-dimensional normed algebra – the quaternions $\mathbb{H}$. Let $a, b, c, d \in \mathbb{R}$. Then

$$\mathbb{H} = \{a + bi + cj + dk : i^2 = j^2 = k^2 = ijk = -1,$$

with $N(a + bi + cj + dk) = (a + bi + cj + dk)(a - bi - cj - dk)$. Note that $ij = k = -ji$, so $\mathbb{H}$ is a *noncommutative* algebra.

December

Graves writes to Hamilton stating that he has constructed an eight-dimensional normed algebra $\mathbb{O}$ he calls the octaves.

January

Graves's January 22 letter to Hamilton contains the details about $\mathbb{O}$.

The Octaves or Octonions $\mathbb{O}$

The set

$$\mathbb{O} = \{a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 : a_n \in \mathbb{R}\}$$

How to multiply real numbers

Multiplication of real numbers is as usual.
If $r \in \mathbb{R}$, then $re_n = e_n r$.

How to multiply the units e_n

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = e_4{}^2 = e_5{}^2 = e_6{}^2 = e_7{}^2 = -1$$

$$e_1 = e_2 e_4 = e_3 e_7 = e_5 e_6 = -e_4 e_2 = -e_7 e_3 = -e_6 e_5$$

$$e_2 = e_3 e_5 = e_4 e_1 = e_6 e_7 = -e_5 e_3 = -e_1 e_4 = -e_7 e_6$$

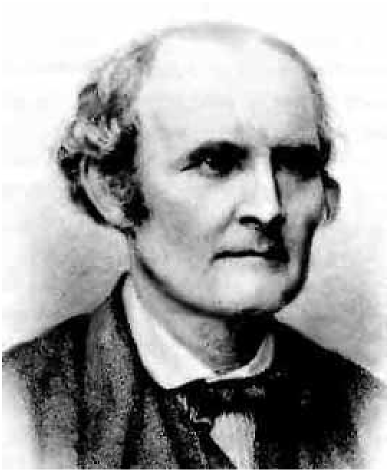
$$e_3 = e_4 e_6 = e_5 e_2 = e_7 e_1 = -e_6 e_4 = -e_2 e_5 = -e_1 e_7$$

$$e_4 = e_5 e_7 = e_6 e_3 = e_1 e_2 = -e_7 e_5 = -e_3 e_6 = -e_2 e_1$$

$$e_5 = e_6 e_1 = e_7 e_4 = e_2 e_3 = -e_1 e_6 = -e_4 e_7 = -e_3 e_2$$

$$e_6 = e_7 e_2 = e_1 e_5 = e_3 e_4 = -e_2 e_7 = -e_5 e_1 = -e_4 e_3$$

$$e_7 = e_1 e_3 = e_2 e_6 = e_4 e_5 = -e_3 e_1 = -e_6 e_2 = -e_5 e_4$$



Arthur Cayley (1821-1895)

1845

Cayley's appendix

Cayley writes a paper about elliptic functions that is practically incomprehensible . . .

. . . except for the one-page appendix, which describes his own version of the octonions.

Seven triples

The appendix includes an explicit mention of seven certain triples of numbers that explain the multiplication.

Remember this, because it's important.

Octonion multiplication and the seven triples

The key to multiplication in $\mathbb{O}$

Cyclically order the elements in the seven triples $(1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 6)$, $(4, 5, 7)$, $(5, 6, 1)$, $(6, 7, 2)$, and $(7, 1, 3)$.

Then $e_a e_b = e_c$ or $e_a e_b = -e_c$ according as a does or does not directly precede b in the unique ordered triple containing a and b.

Nonassociativity

Since

$$(e_1 e_2)e_6 = e_4 e_6 = e_3, \text{ and}$$
$$e_1(e_2 e_6) = e_1 e_7 = -e_3,$$

it follows that $\mathbb{O}$ is not an associative algebra.

Wesley Stoker Barker Woolhouse (1809 - 1893)

Woolhouse and designs

The question, 1844

The Prize Question in the *Lady's and Gentleman's Diary* for 1844, set by the magazine's editor, W. S. B. Woolhouse:

> *Determine the number of combinations that can be made of n symbols, p symbols in each; with this limitation, that no combination of q symbols which may appear in any one of them shall be repeated in any other.*

The question repeated, 1846

The Prize Question in the *Lady's and Gentleman's Diary* for 1846, simplified by Woolhouse:

> *How many triads can be made out of n symbols, so that no pair of symbols shall be comprised more than once amongst them?*

This time, he got an answer – and how!

Kirkman

Thomas P. Kirkman (1806-1895)

1847-1853: Kirkman's wonder years

- 1847: "On a problem in combinations" answers Woolhouse's question, proves that such a system exists if and only if $n = 6m + 1$ or $n = 6m + 3$. (Such systems are called Steiner triple systems.) Describes the system of seven triples for $n = 7$ – the $(7, 3, 1)$ design.
- 1848: Points out that his $(7, 3, 1)$ design is closely related to the algebra of the octonions. (I told you they were important.)
- 1850: Poses the so-called Kirkman Schoolgirls Problem, describes *resolvable* designs. Cayley publishes a solution. Kirkman describes how he found his solution.
- 1852-3: Three papers on combinatorial designs.

The $(7,3,1)$ block design and its discoverers

The design

$B_1 = \{1,2,4\}$
$B_2 = \{2,3,5\}$
$B_3 = \{3,4,6\}$
$B_4 = \{4,5,7\}$
$B_5 = \{5,6,1\}$
$B_6 = \{6,7,2\}$
$B_7 = \{7,1,3\}$

seven elements: $\{1,2,3,4,5,6,7\}$

seven blocks: $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$

each element in three blocks

three elements per block: remember the triples?

each pair of distinct elements in one block together

the smallest nontrivial Steiner triple system

The discoverers: was it ...

- ... John T. Graves in 1843?
- ... Arthur Cayley in 1845?
- ... Thomas Kirkman in 1847?

Answer:

The $(7, 3, 1)$ block design and its discoverers

The design

$$\boxed{\begin{aligned} B_1 &= \{1, 2, 4\} \\ B_2 &= \{2, 3, 5\} \\ B_3 &= \{3, 4, 6\} \\ B_4 &= \{4, 5, 7\} \\ B_5 &= \{5, 6, 1\} \\ B_6 &= \{6, 7, 2\} \\ B_7 &= \{7, 1, 3\} \end{aligned}}$$

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Answer: all three – no evidence that one influenced the others' work.

Jakob Steiner (1796-1863)

Steiner Triple Systems

1853: Steiner's two-page paper asks questions about triple systems and more general combinatorial designs.

He states that if such a system exists on n points, then $n \equiv 1, 3 \bmod 6$, but does not prove it.

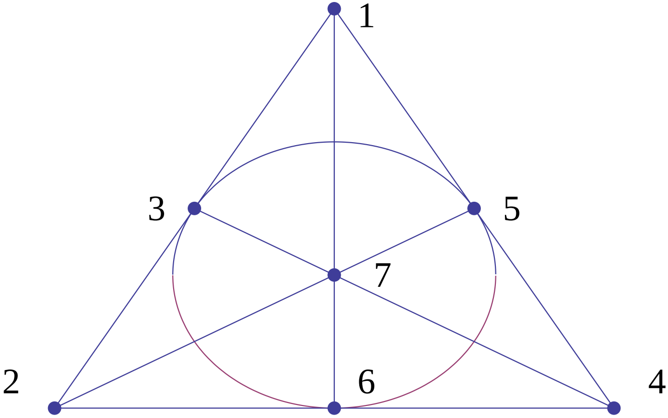
1859: M. Riesz answers Steiner's questions with proof – essentially, Kirkman's 1847 proof.

Riesz' and Steiner's papers appeared in a widely-read journal; Kirkman's papers appeared in more obscure journals.

Fano

Gino Fano (1871-1952)

Gino Fano's finite projective plane of order 2



Projective plane of order $n = 2$: $n + 1 = 3$ points on each line

Whose finite projective plane?

1892: Fano's fundamental paper on projective geometry includes a description of finite projective planes.

1857: Kirkman describes an algebraic method for constructing finite projective planes.

In 1857, Gino Fano's parents had not yet met.

Conclusions

- The geometry of Julius Plücker was intrinsically connected to the combinatorics of Thomas Kirkman.
- The combinatorics of Kirkman anticipated Jakob Steiner's questions.
- The work of Gino Fano in 1892 provided a link from projective geometry to both Plücker and the realm of Steiner triple systems.
- Fano's work provided a link via $(7, 3, 1)$ to the combinatorics of Kirkman, as well as to Hamilton, Graves and Cayley via the normed algebra of octonions ...
- ... and consequently to the Diophantine problem of products of sums of squares.

Don't go away. We'll be right back.

THANK YOU!