A CLASSIFICATION OF QUADRATIC ROOK POLYNOMIALS

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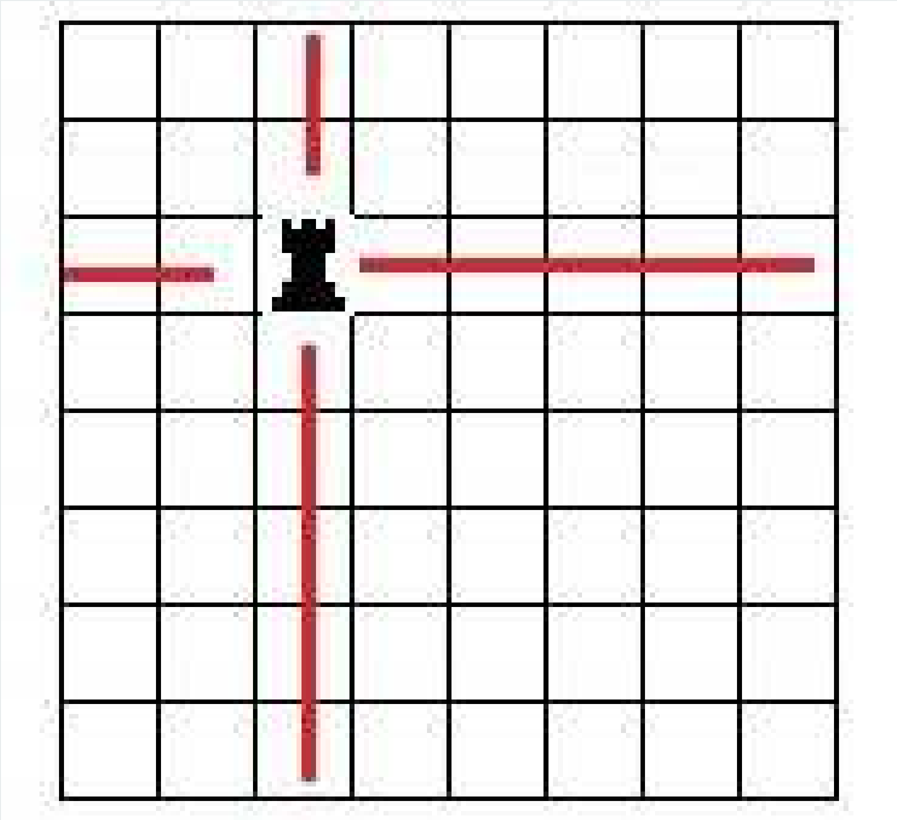
TOPICS TO BE DISCUSSED

- Rook Theory and relevant definitions
- General examples
- Our Problem
- Solution

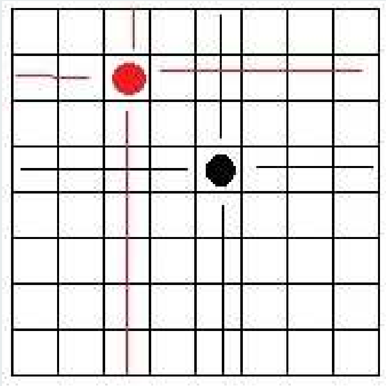
ROOKS

- In chess, a rook can attack in any square in its row or column

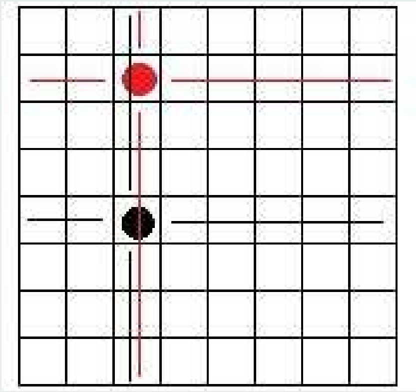


ATTACKING VS. NON-ATTACKING ROOKS

- Non-attacking rooks



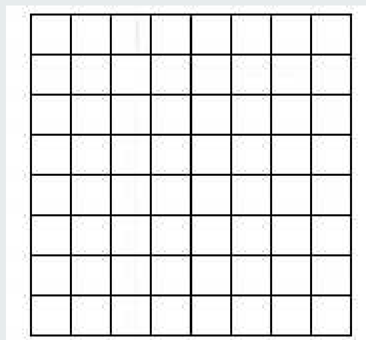
- Attacking rooks



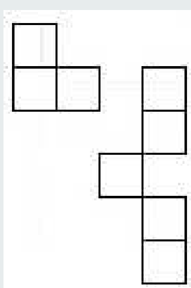
- We will be focusing on non-attacking rooks

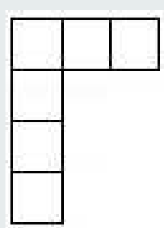
BOARDS AND GENERALIZED BOARDS

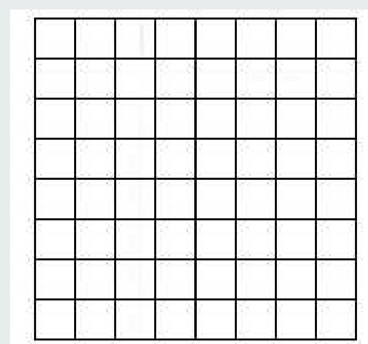
- Board- a square n x n chessboard

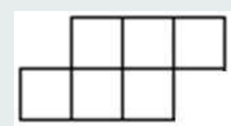


- Generalized Board- any subset of a board









ROOK NUMBERS

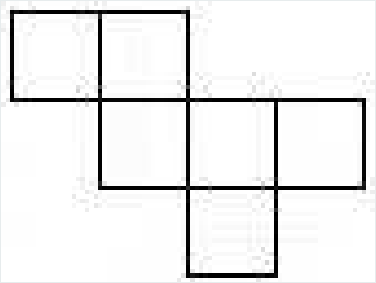
- The k^{th} rook number $r_k(B)$ counts the number of ways to place k non-attacking rooks on a generalized board B
- We will often denote $r_k(B)$ as r_k when B is clear
- r_0 is always 1
  - Only one way to place 0 rooks on a board

ROOK NUMBERS

- r_1 is the number of squares on B
  - The rook can be placed in any square since there will be no other rook for it to attack

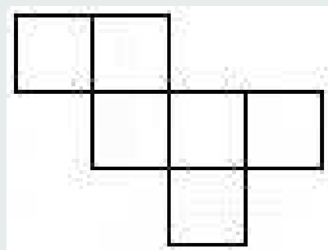
- Once we attain $r_k = 0$, we will always have $r_{k+1}, r_{k+2}, \ldots = 0$

- $r_k = 0$ when k> the number of rows or columns in B



EXAMPLE OF ROOK NUMBERS

- Consider the following generalized board:

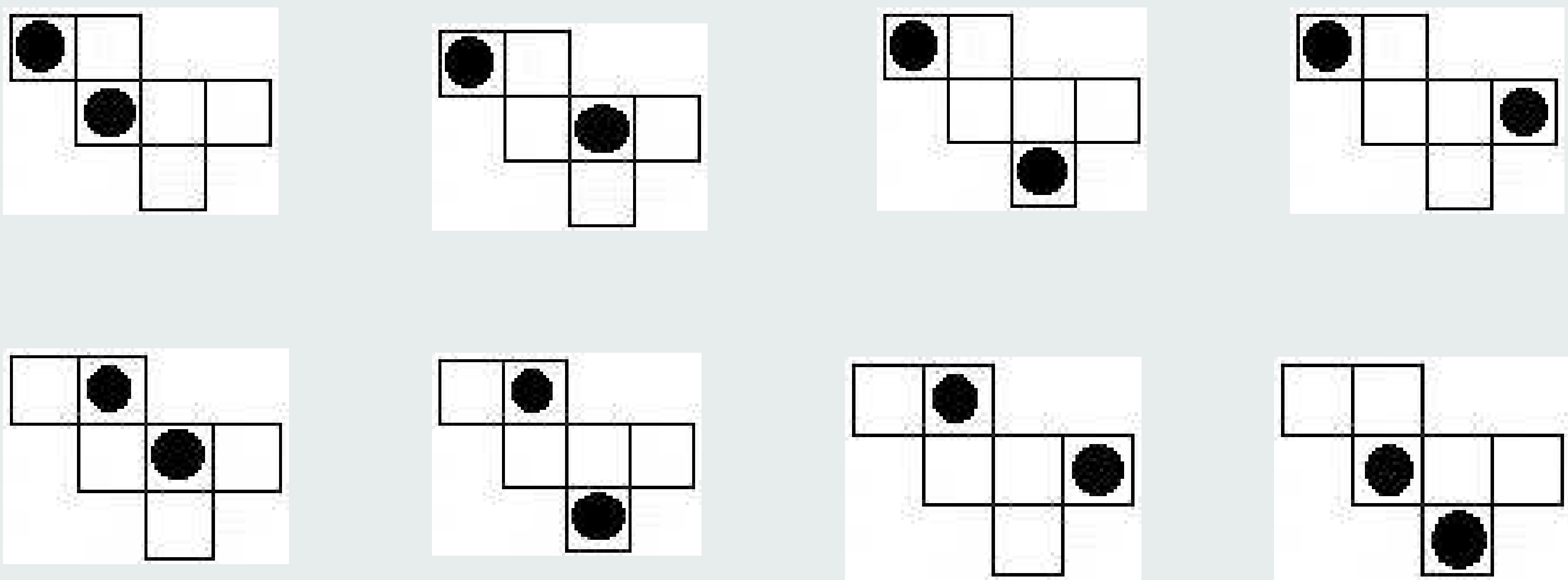


- $r_0 = 1$, $r_1 = 6$

r_2

- There are 8 ways to place 2 rooks on the generalized board so that they are non-attacking.

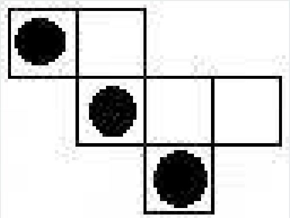
- $r_2 = 8$

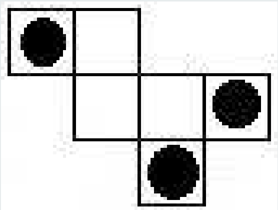


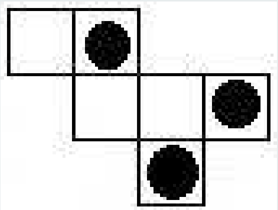
r_3

- There are 3 ways to place 3 rooks on the generalized board so they are non-attacking

- $r_3 = 3$



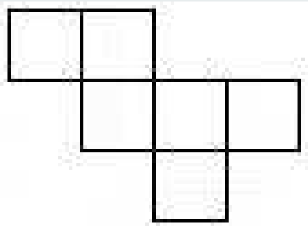




r_4 , r_5 , …

- There are only 3 rows on this generalized board, therefore
  $r_4 = 0$

- thus, $r_{5,}$ $r_{6,\ldots} = 0$



ROOK POLYNOMIAL

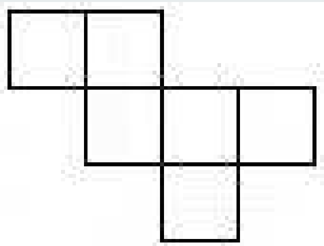
- We can construct a polynomial which keeps track of all of the rook numbers of a generalized board at once

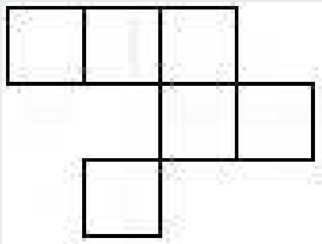
- The r_k's are the coefficients of the x^k terms

$$r_0 + r_1x + r_2x^2 + \ldots + r_{k-1}x^{k-1} + r_kx^k$$

ROOK POLYNOMIAL EXAMPLE

- Considering the rook numbers from our previous example:
  - $r_0 = 1$, $r_1 = 6$, $r_2 = 8$, $r_3 = 3$, $r_4 = 0$, $r_{5,} r_{6,\ldots} = 0$
  - $1 + 6x + 8x^2 + 3x^3$
- The generalized board on the left is from our example.
- Doing some work, we could show that the generalized board on the right also has the same rook polynomial
- Thus, rook polynomials are not unique to a single generalized board





OUR PROBLEM

- Classify all quadratic polynomials which are the rook polynomial for some generalized board B

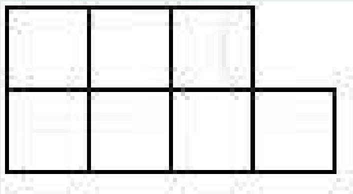
- Know $r_0 = 1$, r_1 = number of squares of B
  - the form of our polynomials will be: $1 + r_1x + r_2x^2$

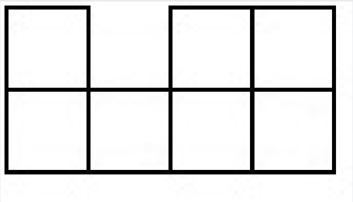
- Since r_1 can be any positive integer greater than 1 ($r_1=1$ would lead to a linear rook polynomial), we must find all possible r_2's such that $r_3 = 0$.
  - If $r_3 \neq 0$, our rook polynomial could be cubic or of higher degree.

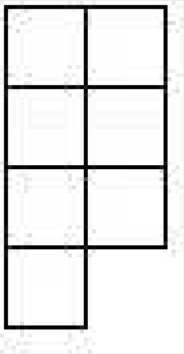
OUR PROBLEM

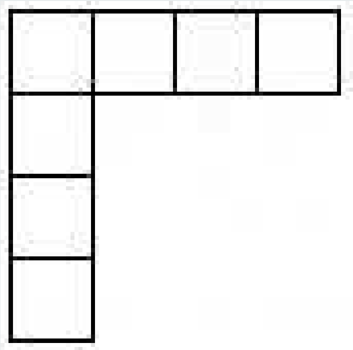
- Recall that a rook polynomial is not unique to a single generalized board.

- Consider the generalized boards below. Each has the same rook polynomial.





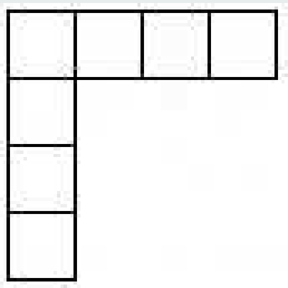




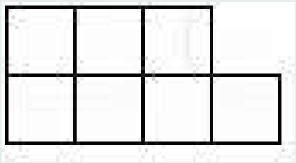
OUR PROBLEM

- Clearly, if the generalized board is contained within 2 rows, we will have $r_3 = 0$.
  - Is the converse true? For our purposes… YES

- We proved that for $r_3 = 0$, the generalized board must either be contained within 2 rows or have the L-shaped form seen below.

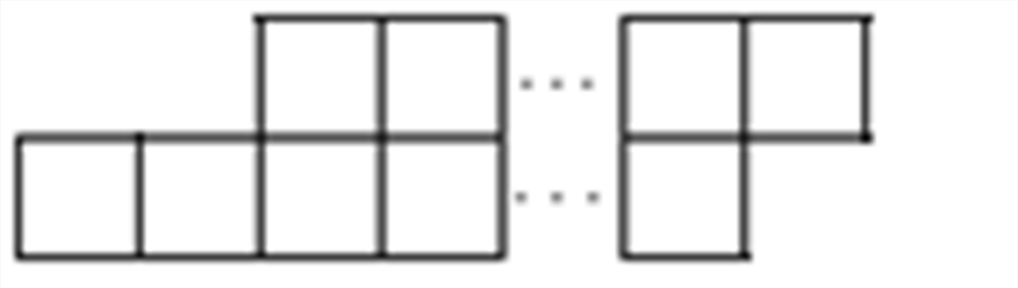


  - Recall that the L-shaped board has the same rook polynomial as the board below.



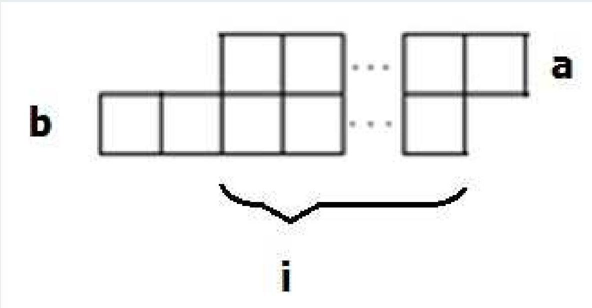
OUR PROBLEM

- Taking into account the equivalences and the requirement of $r_3 = 0$ we found that it suffices to consider generalized boards which meet the following conditions
  - lie within two rows of a board
  - have spaces which lie consecutive within each row



OBTAINING POSSIBLE r_2'S GIVEN r_1

- Let r_1 represent the number of squares of the generalized board.
- Consider each pair of integers *a* and *b*, where $r_1=a+b$ and $a \le b$.
  - Note that $b=r_1 - a$
- Then *a* and *b* can be arranged such that *a* squares lie consecutively in one row and *b* squares lie consecutively in the next row.
- Let i= the number of columns where the two rows overlap

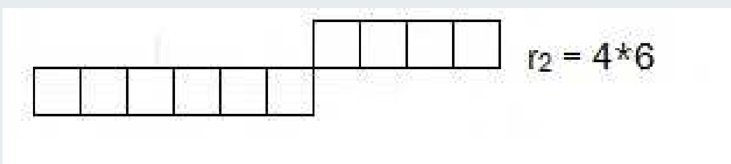


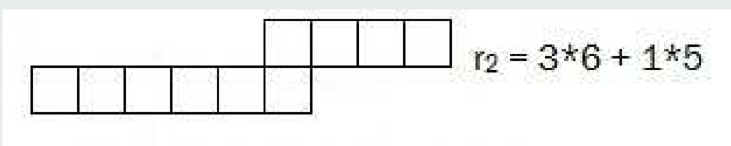
EXAMPLE FOR FINDING r2

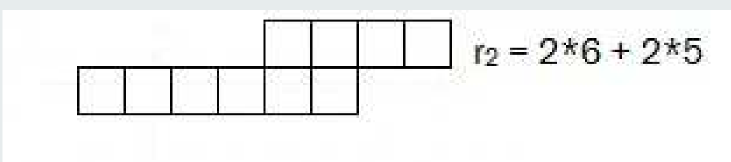
- Consider r_1=10

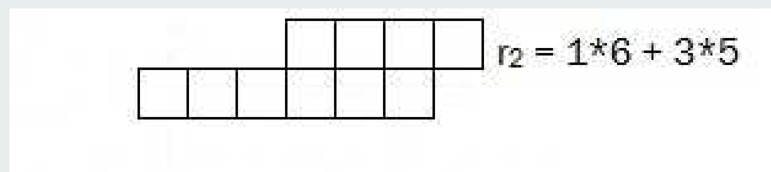
- The possible pairs for ***a*** and ***b*** are:
  - 1, 9; 2,8; 3,7; 4,6; 5,5

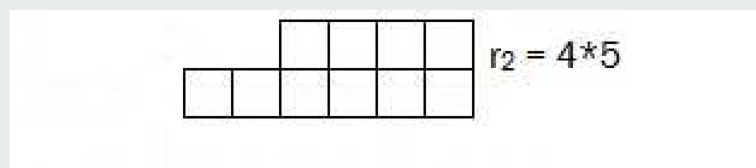
- Let's look at a = 4 and b = 6







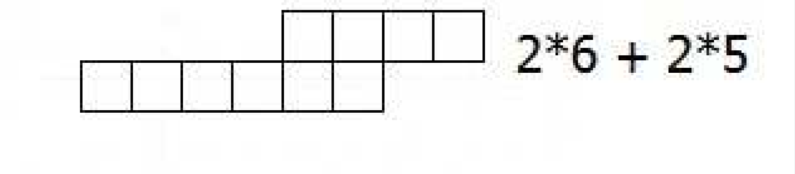




- This can be done for all of the pairs listed above

CREATING A FORMULA

- Consider again the generalized board below
  - $r_2 = 2*6 + 2*5$
  - $r_2 = (4-2)*6 + 2*(6-1)$
  - $r_2 = (a-i)*b + i*(b-1)$



FORMULA FOR r_2 GIVEN r_1

- When we simplify, we will see
  - $r_2 = ab - ib + ib - i$
  - $r_2 = ab - i$
    - Recall $b = r_1 - a$
  - $r_2 = a(r_1 - a) - i$

- Given a first rook number of r_1 every r_2 will have the form
  - $r_2 = a(r_1 - a) - i$, $0 \leq i \leq a$

EXAMPLE

- Let r_1=10. The pairs of *a* and *b* for r_1 are as follows:
  - 1, 9; 2, 8; 3, 7; 4, 6; 5, 5.

- Apply the formula $r_2=a(r_1 - a) - i$ for each pair.

(1)(9) – 0 = 9	(3)(7) – 0 = 21	(4)(6) – 0 = 24	(5)(5) – 0 = 25
(1)(9) – 1 = 8	(3)(7) – 1 = 20	(4)(6) – 1 = 23	(5)(5) – 1 = 24
	(3)(7) – 2 = 19	(4)(6) – 2 = 22	(5)(5) – 2 = 23
(2)(8) – 0 = 16	(3)(7) – 3 = 18	(4)(6) – 3 = 21	(5)(5) – 3 = 22
(2)(8) – 1 = 15		(4)(6) – 4 = 20	(5)(5) – 4 = 21
(2)(8) – 2 = 14			(5)(5) – 5 = 20

- A generalized board with 10 squares can obtain every value for r_2 between 8 and 25 except for 10, 11, 12, 13, and 17.

LIST OF ALL QUADRATIC ROOK POLYNOMIALS WHERE r_1=10

- $1 + 10x + 8x^2$
- $1 + 10x + 9x^2$
- $1 + 10x + 14x^2$
- $1 + 10x + 15x^2$
- $1 + 10x + 16x^2$
- $1 + 10x + 18x^2$
- $1 + 10x + 19x^2$
- $1 + 10x + 20x^2$
- $1 + 10x + 21x^2$
- $1 + 10x + 22x^2$
- $1 + 10x + 23x^2$
- $1 + 10x + 24x^2$
- $1 + 10x + 25x^2$

- This can be done for any value of r_1

CONCLUSION

- Thus, we can construct all possible quadratic rook polynomials using the formula below:
  - $1 + r_1x + [r_1(a - r_1) - i]x^2$
    - For positive integers r_1, a, i
    - where $r_1 > 1$
    - where $1 \leq a \leq [r_1/2]$
    - where $0 \leq i \leq a$